Deformation Quantization and Symmetries

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Autumn School 2019 : Deformations and Rigidity in Algebra, Geometry and Analysis

Programme

Introduction to the concept of deformation quantization (existence, classification and representation results for formal star products).

Notion of formal star products with symmetries; one has a Lie group action (or a Lie algebra action) compatible with the classical Poisson structure, and one wants to consider star products such that the Lie group acts by automorphisms (or the Lie algebra acts by derivations). We recall in particular the link between left invariant star products on Lie groups and Drinfeld twists, and the notion of universal deformation formulas.

Quantum moment map : Classically, symmetries are particularly interesting when they are implemented by a moment map. We give indications to build a corresponding quantum version. Concerning links between representation theory and the quantization of an orbit of a group in the dual of its Lie algebra, we recall how some star products yield an adapted Fourier transform.

Quantum reduction : reduction is a construction in classical mechanics with symmetries which allows to reduce the dimension of the manifold; we describe one of the various quantum analogues which have been considered in the framework of formal deformation quantization.

Considerations about convergence of star products.

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Why Quantization?

Quantum theory provides a description of nature which is more fundamental than classical theory. We shall consider here only non relativistic descriptions.

Why are we interested in quantization, nature being quantum?

- Giving a quantum description a priori of a physical system is difficult, and the classical description is often easier to obtain; hence one often uses the classical description as a starting point to find a quantum description.

- Any given physical theory remains valid within a range of measurements, so that any modified theory should give the same results in the initial range.

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Classical Mechanics

Classical mechanics, in its Hamiltonian formulation on the motion space, has for framework a symplectic manifold (or more generally a Poisson manifold).

A **Poisson bracket** defined on the space of real valued smooth functions on a manifold M, is a $\mathbb{R}$- bilinear map on $C^\infty(M)$, $(u,v)\mapsto\{u,v\}$ such that for any $u,v,w\in C^\infty(M)$:

- $\{u,v\}=-\{v,u\}$ (skewsymmetry),
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- $\{\{u,v\},w\}+\{\{v,w\},u\}+\{\{w,u\},v\}=0$ (Jacobi's identity).

A Poisson bracket is given in terms of a contravariant skew symmetric 2-tensor P on M, the **Poisson tensor**, via $\{u,v\}=P(du\wedge dv)=P^{rs}\partial_r u\partial_s v \quad$ (so that $[P,P]=0$.)
(M,ω) is **symplectic** if ω is a non degenerate closed 2-form; one then defines $\{u,v\}=-\omega(X_u,X_v)$ where X_u is the Hamiltonian vector field associated to u, i.e. $\iota(X_u)\omega=du$.

On $(\mathbb{R}^{2n},dp_i\wedge dq^i)$, the bracket is $\{f,g\}=\partial_{q^i}f\partial_{p_i}g-\partial_{p_i}f\partial_{q^i}g$.

The motion space is in general the quotient of the evolution space by the motion.

Observables are families of smooth functions on that manifold M.
The dynamics is defined in terms of a Hamiltonian $H\in C^\infty(M)$ and the time evolution of an observable $\{f_t\}_{t\in I}; f_t\in C^\infty(M)$ is governed by the equation :

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Quantum mechanics, in its usual Heisenberg's formulation, has for framework a Hilbert space (states are rays in that space).
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The dynamics is defined in terms of a Hamiltonian H, which is a selfadjoint operator, and the time evolution of an observable $\{A_t\}_t$ is governed by the equation :

$$\frac{dA_t}{dt} = \frac{i}{\hbar}[H, A_t].$$

We have seen that classical mechanics has for framework a symplectic or Poisson manifold.
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There is no correspondence defined on all smooth functions on M so that

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when one puts an irreducibility requirement which is necessary not to violate Heisenberg's principle. More precisely, Van Hove proved that there is no irreducible representation of the Heisenberg algebra, viewed as the algebra of constants and linear functions on $\mathbb{R}^{2n}$ endowed with the Poisson braket, which extends to a representation of the algebra of polynomials on $\mathbb{R}^{2n}$.

Flato, Lichnerowicz and Sternheimer introduced Deformation Quantization where they " suggest that quantisation be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables."

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Deformation

" La richesse d'un concept scientifique se mesure à sa puissance de déformation."

La formation de l'esprit scientifique - Gaston Bachelard

Formal Deformation quantization - Star Products

A **star product** on a Poisson manifold (M, P) is a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\nu]] : \qquad (u, v) \mapsto u \star v = u \star_\nu v := \sum_{r \geq 0} \nu^r C_r(u, v)$$

such that :
(a) when the map is extended ν-linearly (and continuously in the ν-adic topology) to $C^\infty(M)[[\nu]] \times C^\infty(M)[[\nu]]$ it is formally associative:

$$(u \star v) \star w = u \star (v \star w);$$

(b) $C_0(u, v) = uv =: \mu(u, v)$,
(c) $C_1(u, v) - C_1(v, u) = \{u, v\} = P(du \wedge dv)$;
(d) $1 \star u = u \star 1 = u$;

the C_r's are bidifferential operators on M (it is then a **differential star product**).

When each C_r is of order $\leq r$ in each argument, $\star$ is called **natural** .

If $\overline{f \star g} = \overline{g} \star \overline{f}$ for any purely imaginary $\nu = i\lambda$, $\star$ is called **Hermitian**.

Example 1: Moyal-Weyl $\star$-product and the Weyl algebra

Let P be a Poisson structure on $V=\mathbb{R}^m$ with constant coefficients:

$$P=\sum_{i,j}P^{ij}\partial_i\wedge\partial_j,\quad P^{ij}=-P^{ji}\in\mathbb{R}$$

The Weyl-Moyal $\star$ product is

$$(u\star_M v)(z)=\exp\left(\frac{\nu}{2}P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=z}.$$

Associativity follows from the fact that $\partial_{t^k}(u\star_M v)(t)=(\partial_{x^k}+\partial_{y^k})\exp\left(\frac{\nu}{2}P^{rs}\partial_{x^r}\partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=t}$:

$$\begin{aligned}((u\star_M v)\star_M w)(x') &= \exp\left(\frac{\nu}{2}P^{rs}\partial_{t^r}\partial_{z^s}\right)((u\star_M v)(t)w(z))\Big|_{t=z=x'}\\ &= \exp\left(\frac{\nu}{2}P^{rs}(\partial_{x^r}+\partial_{y^r})\partial_{z^s}\right)\exp\left(\frac{\nu}{2}P^{r's'}\partial_{x^{r'}}\partial_{y^{s'}}\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'}\\ &= \exp\left(\frac{\nu}{2}P^{rs}(\partial_{x^r}\partial_{z^s}+\partial_{y^r}\partial_{z^s}+\partial_{x^r}\partial_{y^s})\right)((u(x)v(y))w(z))\Big|_{x=y=z=x'}=(u\star_M(v\star_M w)(x').\end{aligned}$$

When P is non degenerate, $(S(V^*)[\nu],\star_M)$ is called **the Weyl algebra** .

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Associativity follows from the fact that $\partial_{t^k}(u \star_M v)(t) = (\partial_{x^k} + \partial_{y^k}) \exp\left(\frac{\nu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=t}$:

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When P is non degenerate, $(S(V^*)[\nu], \star_M)$ is called **the Weyl algebra** .

Example 1: Moyal-Weyl $\star$-product and the Weyl algebra

Let P be a Poisson structure on $V = \mathbb{R}^m$ with constant coefficients:

$$P = \sum_{i,j} P^{ij} \partial_i \wedge \partial_j, \quad P^{ij} = -P^{ji} \in \mathbb{R}$$

The Weyl-Moyal $\star$ product is

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Relation to Weyl's quantization

For the usual quantization of $\mathbb{R}^{2n}$ with the canonical Poisson bracket [in coordinates $\{ q^i, p_i; 1 \leq i \leq n \}$ $\{u, v\} = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial q^j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q^j} \right)$]

the *Weyl ordering* yields a bijection $\mathcal{Q}_{Weyl}$ between polynomials on $\mathbb{R}^{2n}$, $\mathbb{C}[p_i, q^j]$ and the space of differential operators with complex polynomial coefficients $D_{polyn}(\mathbb{R}^n)$:

$$\mathcal{Q}_{Weyl}(1) = \mathrm{Id}, \qquad \mathcal{Q}_{Weyl}(q^i) := Q^i := q^i\cdot, \qquad \mathcal{Q}_{Weyl}(p_i) := P_i = i\hbar \frac{\partial}{\partial q^i}$$

and to a polynomial in $p's$ and $q's$ the corresp. totally symmetrized polynomial in Q^i and P_j:

$$\mathcal{Q}_{Weyl}(q^1(p^1)^2) = \frac{1}{3}(Q^1(P^1)^2 + P^1Q^1P^1 + (P^1)^2Q^1).$$

Then

$$\begin{aligned} f *_W g : &= \mathcal{Q}_{Weyl}^{-1}\left(\mathcal{Q}_{Weyl}(f) \circ \mathcal{Q}_{Weyl}(g)\right) \\ &= f.g + \frac{i\hbar}{2}\{f, g\} + O(\hbar^2) = f \star_M g|_{\nu = i\hbar}. \end{aligned} \quad (1)$$

Hence the Moyal star product is related to the composition of operators via Weyl's quantisation of polynomials on $\mathbb{R}^{2n}$.

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Existence of a star product on any symplectic manifold

On any symplectic manifold (M, ω) there exists a differential star product (1983, De Wilde and Lecomte). In 1985 and 1994, Fedosov gave a recursive construction when one has chosen a symplectic connection and a sequence of closed 2-forms $\tilde{\Omega} = \sum_{k \geq 1} \nu^k \omega_k$ on M (a symplectic connection is a linear torsion free connection ∇ such that $\nabla\omega = 0$. Such a connection exists on any symplectic manifold, but is not unique):

it is obtained by identifying $C^\infty(M)[[\nu]]$ with an algebra of flat sections of the Weyl bundle

$\mathcal{W} = F(M) \times_{Sp(V,\Omega),\rho} W$ endowed with a flat connection D built from ∇ et $\tilde{\Omega}$.

$F(M)$ is the bundle of symplectic frames (a symplectic frame at $x \in M$ is a linear sympl. iso. $\xi_x : (V, \Omega) \to (T_xM, \omega_x)$). $F(M)$ is a principal $Sp(V, \Omega)$-bundle over M.

V^* is endowed with the constant Poisson structure $P = \Omega^{-1}$, and one puts a grading on the Weyl algebra $S(V^*)[\nu]$, assigning the degree 1 to $y \in V$ and the degree 2 to ν. Recall that the product is given by $(u \star_M v)(z) = \exp\left(\frac{\nu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right) (u(x)v(y))\big|_{x=y=z}$,
The **formal Weyl algebra** W is the completion in that grading.

ρ is the natural representation of the symplectic group $Sp(V, \Omega)$ on W (for any $B \in sp(V, \Omega)$, $\rho_*(B)a = \frac{-1}{\nu}[\overline{B}, a]$ where $[a, b] := (a \star_M b) - (b \star_M a)$ for any $a, b \in W$ and $\overline{B} = \frac{1}{2} \sum_{ijr} \Omega_{ri} B^r_j y^i y^j$); it acts by automorphisms of $\star_M$, so the Weyl bundle is a bundle of algebras, the fiber product being $\star_M$.

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Fedosov's construction on any symplectic manifold

The symplectic connection induces a connection ∂ in $\mathcal{W}$: $\partial a = da - \frac{1}{\nu}[\frac{1}{2}\omega_{ki}\Gamma^k_{rj}y^iy^j, a]$.

Deform it into $Da = \partial a - \delta(a) - \frac{1}{\nu}[r,a]$ where $\delta(a) = \frac{1}{\nu}\left[-\omega_{ij}y^i dx^j, a\right] = \sum_k dx^k \wedge \frac{\partial a}{\partial y^k}$, with $r \in \Gamma(\mathcal{W}\otimes\Lambda^1)$.
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A r so that $\delta r = -\overline{R} + \partial r - \frac{1}{\nu}r^2 + \tilde{\Omega}$ is given inductively by $r = -\hat{\delta}\overline{R} + \hat{\delta}\partial r - \frac{1}{\nu}\hat{\delta}r^2 + \hat{\delta}\tilde{\Omega}$ where, writing $a = \sum_{p\geq 0, q\geq 0} a_{pq} = \sum_{2k+p\geq 0, q\geq 0} \nu^k a_{k,i_1,\dots,i_p,j_1,\dots,j_q} y^{i_1}\dots y^{i_p}\, dx^{j_1}\wedge\dots\wedge dx^{j_q}$, for any $a \in \Gamma(\mathcal{W}\otimes\Lambda^q)$,

$$\hat{\delta}(a_{pq}) = \frac{1}{p+q}\sum_k y^k i(\frac{\partial}{\partial x^k})a_{pq} \text{ if } \ \mathrm{p+q>0} \text{ and } 0 \text{ if } \ \mathrm{p+q=0}$$

.

A flat section of $\mathcal{W}$ is given inductively by $a = \hat{\delta}\left(\partial a - \frac{1}{\nu}[r,a]\right) + a_{00}$.
It is determined by $a_{00} \in \mathbb{C}^\infty(M)[[\nu]]$ and is denoted $Q(a_{00})$.

The Fedosov's star product $*_{\nabla,\Omega}$ is then obtained by $u \star_{\nabla,\Omega} v := (Q(u)\star_M Q(v))_{00}$.

Equivalence of star products

Given a star product $\star$ and any series $T = \sum_{r\geq 1} \nu^r T_r$ of linear operators on $\mathcal{A} = C^\infty(M)$, one can build another star product denoted $\star' := T \bullet \star$ via

$$u \star' v := e^T \left(e^{-T} u \star e^{-T} v \right). \tag{2}$$

Two star products $\star$ and $\star'$ are said to be **equivalent** if there exists a series T such that equation (2) is satisfied.
If the star products are differential and equivalent, the equivalence can be defined by a series of differential operators.

The classification of star products up to equivalence on symplectic manifolds was obtained by Nest-Tsygan, Deligne , and Bertelson-Cahen-Gutt :
Any star product on a symplectic manifold is equivalent to a Fedosov's one and its equivalence class is parametrised by the element in $H^2(M;\mathbb{R})[[\nu]]$ given by the series $[\tilde{\Omega}]$ of de Rham classes of the closed 2-forms used in the construction.

Fedosov had obtained the classification of star products obtained by his construction, and

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Star products for linear Poisson structures

An explicit construction of star product was known for linear Poisson structure, i.e. on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with the Poisson structure defined by

$$P_\xi(X,Y) :=< \xi, [X,Y] >, \qquad \xi \in \mathfrak{g}^*, X, Y \in \mathfrak{g} \simeq T^*_\xi \mathfrak{g}^*,$$

using the fact that polynomials on $\mathfrak{g}^*$ identify with the symmetric algebra $S(\mathfrak{g})$ which in turns is in bijection with the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ via

$$\sigma : S(\mathfrak{g}) \to \mathcal{U} \quad X_1 \dots X_k \mapsto \frac{1}{k!} \sum_{\rho \in S_k} X_{\rho(1)} \circ \dots \circ X_{\rho(k)}.$$

Pulling back the associative structure of $\mathcal{U}(\mathfrak{g})$ to the space of polynomials on $\mathfrak{g}^*$ yields a differential star product

$U(\mathfrak{g}) = \oplus_{n \geq 0} U_n$ where $U_n := \sigma(S^n(\mathfrak{g}))$ and we decompose an element $u \in U(\mathfrak{g})$ accordingly $u = \sum u_n$. For $P \in S^p(\mathfrak{g})$ and $Q \in S^q(\mathfrak{g})$:

$$P * Q = \sum_{n \geq 0} (\nu)^n \sigma^{-1}((\sigma(P) \circ \sigma(Q))_{p+q-n}). \qquad (3)$$

This star product is characterised by

$$X * X_1 \dots X_k = X X_1 \dots X_k + \sum_{j=1}^{k} \frac{(-1)^j}{j!} \nu^j Bj \,[[..[X, X_{r_1}], \dots], X_{r_j}] X_1 \dots \widehat{X_{r_1}} \dots \widehat{X_{r_j}} \dots X_k$$

where B_j are the Bernouilli numbers.

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Star products on a Poisson manifold

A proof by Masmoudi of existence of a star product on a regular Poisson manifold quickly followed the proofs in the symplectic setting.

For general Poisson manifolds, existence and classification of star products were given by Kontsevich in 1995 :
The set of equivalence classes of differential star products on a Poisson manifold (M, P) coincides with the set of equivalence classes of Poisson deformations of P:

$$P_\nu = P\nu + P_2\nu^2 + \cdots \in \nu\Gamma(X, \Lambda^2 T_X)[[\nu]], \text{ such that } [P_\nu, P_\nu]_S = 0,$$

where equivalence of Poisson deformations is defined via the action of a formal vector field on M, $X = \sum_{r\geq 1} \nu^r X_r$, via $\{u, v\}' := e^X \left\{e^{-X}u, e^{-X}v\right\}$.

Remark that in the symplectic framework, this result coincides with the previous one. Indeed any Poisson deformation P_ν of the Poisson bracket P on a symplectic manifold (M, ω) is of the form P^Ω for a series $\Omega = \omega + \sum_{k\geq 1} \nu^k \omega_k$ where the ω_k are closed 2-forms, through

$$P^\Omega(du, dv) = -\Omega(X_u^\Omega, X_v^\Omega), \qquad \text{with } \; X_u^\Omega \in \Gamma(TM)[[\nu]] \text{ defined by } \quad i(X_u^\Omega)\Omega = du.$$

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Formal Deformation and DGLA

We briefly sketch how Kontsevich's theorem is a consequence of his formality theorem. A general yoga sees any deformation theory encoded in a differential graded Lie algebra structure.

A **differential graded Lie algebra** (briefly DGLA) $(\mathfrak{g}, [\ ,\ ], d)$ is a $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}^i, [\ ,\ ]$ with $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$,together with a differential $d\colon \mathfrak{g} \to \mathfrak{g}$, i.e. a graded derivation of degree 1 ($d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}$, $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$) so that $d \circ d = 0$.

A **deformation** is a Maurer-Cartan element, i.e. a $C \in \nu\mathfrak{g}^1[[\nu]]$ so that $dC - \frac{1}{2}[C, C] = 0$.

Equivalence of deformations is obtained through the action of the group $\exp \nu\mathfrak{g}^0[[\nu]]$, the infinitesimal action of a $T \in \nu\mathfrak{g}^0[[\nu]]$ being $T \cdot C := -dT + [T, C]$..

To express star products in that framework, consider **the DGLA of polydifferential operators.**

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The Hochschild DGLA associated to an associative algebra

Let $(\mathcal{A}, \mu)$ be an associative algebra with unit on a field $\mathbb{K}$.
Consider the **Hochschild complex** of multilinear maps from $\mathcal{A}$ to itself:

$$\mathcal{C}(\mathcal{A}) := \sum_{i=-1}^{\infty} \mathcal{C}^i \quad \text{with} \quad \mathcal{C}^i := \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}^{\otimes(i+1)}, \mathcal{A});$$

remark that the degree is shifted by one; the degree $|A|$ of a $(p+1)$–linear map A is equal to p.

For $A_1 \in \mathcal{C}^{m_1}$, $A_2 \in \mathcal{C}^{m_2}$, define:

$(A_1 \circ A_2)1(f_1, \ldots, f_{m_1+m_2+1}) := \sum_{j=1}^{m_1}(-1)^{(m_2)(j-1)} A_1(f_1, \ldots, f_{j-1}, A_2(f_j, \ldots, f_{j+m_2}), f_{j+m_2+1}, \ldots, f_{m_1+m_2+1})$.

The **Gerstenhaber bracket** is defined by $[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{m_1 m_2} A_2 \circ A_1$. It gives $\mathcal{C}$ the structure of a graded Lie algebra.

An element $M \in \mathcal{C}^1$ defines an associative product iff $[M, M]_G = 0$.

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Star products and DGLA of polydifferential operators

Here we consider $\mathcal{A} = C^\infty(M)$, and we deal with the subalgebra of $\mathcal{C}(\mathcal{A})$ consisting of multidifferential operators

$$\mathcal{D}_{poly}(M) := \bigoplus \mathcal{D}^i_{poly}(M) \quad \text{with} \quad \mathcal{D}^i_{poly}(M)$$

the set of multi differential operators acting on $i+1$ smooth functions on M and vanishing on constants.

$\mathcal{D}_{poly}(M)$ is closed under the Gerstenhaber bracket and under the differential d_μ, so that

$$(\mathcal{D}_{poly}(M), [\ ,\ ]_G, d_{\mathcal{D}} := d_\mu|_{\mathcal{D}}) \qquad \text{is a DGLA.}$$

A $\star$-product is given by a series of bidifferential operators $\star = \mu + C$ with $C \in \nu\mathcal{D}^1_{poly}(M)[[\nu]]$ a Maurer-Cartan element of the DGLA $(\mathcal{D}_{poly}(M), [\ ,\ ]_G, d_{\mathcal{D}})$; indeed the associativity $[\mu + C, \mu + C]_G = 0$ is equivalent to $d_D C - \frac{1}{2}[C, C]_G = 0$.

Equivalence of star products is given by the action of e^T with $T \in \nu\mathcal{D}^0_{poly}(M)[[\nu]]$ via : $\mu + C' = (\exp[T,\ ]_G)(\mu + C)$; the infinitesimal action is $T \cdot C := -d_{\mathcal{D}} T + [T, C]_G$.

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Poisson deformations and the DGLA of polyvectorfields

Any DGLA $(\mathfrak{g}, [\ ,\ ], d)$ has a cohomology complex defined by

$$H^i(\mathfrak{g}) := \mathrm{Ker}(d\colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}) \Big/ \mathrm{Im}(d\colon \mathfrak{g}^{i-1} \to \mathfrak{g}^i).$$

The set $H := \bigoplus_i H^i(\mathfrak{g})$ inherits the structure of a graded Lie algebra, defined by: $[|a|, |b|]_H := |[a, b]|$ where $|a| \in \mathcal{H}$ denote the equivalence classes of a closed element $a \in \mathfrak{g}$. Then $(H, [\ ,\ ]_H, 0)$ is a DGLA (with zero differential).

Thm [Vey] Every $C \in \mathcal{D}^p_{poly}(M)$ such that $d_{\mathcal{D}}(C) = 0$ is the sum of the coboundary of a $B \in \mathcal{D}^{p-1}_{poly}(M)$ and a 1-differential skewsymmetric p-cocycle A:

$$H^p(\mathcal{D}_{poly}(M)) = HH^p_{\mathrm{diff}}(C^\infty(M), C^\infty(M)) = \Gamma(\Lambda^{p+1}TM) =: \mathcal{T}^p_{poly}(M).$$

The bracket induced on $\mathcal{T}_{poly}(M)$ is -(up to a sign $[T_1, T_2]_{\mathcal{T}} := -[T_2, T_1]_S$) - the **Schouten-Nijenhuis bracket** defined by extending the usual Lie bracket of vector fields

$$[X_1 \wedge \ldots \wedge X_k, Y_1 \wedge \ldots \wedge Y_l]_S = \sum_{r=1}^k \sum_{s=1}^l (-1)^{r+s}[X_r, X_s] X_1 \wedge \ldots \widehat{X_r} \wedge \ldots \wedge X_k \wedge Y_1 \wedge \ldots \widehat{Y_s} \wedge \ldots \wedge Y_l.$$

$(\mathcal{T}_{poly}(M), [\ ,\ ]_{\mathcal{T}}, 0)$ is a DGLA.

A $P \in \nu\mathcal{T}^1_{poly}(M)[[\nu]]$ defines a formal Poisson structure on M iff $d_{\mathcal{T}} P - \frac{1}{2}[P, P]'_S = 0$.

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The set $H := \bigoplus_i H^i(\mathfrak{g})$ inherits the structure of a graded Lie algebra, defined by: $[|a|, |b|]_H := |[a, b]|$ where $|a| \in \mathcal{H}$ denote the equivalence classes of a closed element $a \in \mathfrak{g}$. Then $(H, [\,,\,]_H, 0)$ is a DGLA (with zero differential).

Thm [Vey] Every $C \in \mathcal{D}^p_{poly}(M)$ such that $d_{\mathcal{D}}(C) = 0$ is the sum of the coboundary of a $B \in \mathcal{D}^{p-1}_{poly}(M)$ and a 1-differential skewsymmetric p-cocycle A:

$$H^p(\mathcal{D}_{poly}(M)) = HH^p_{\mathrm{diff}}(C^\infty(M), C^\infty(M)) = \Gamma(\Lambda^{p+1} TM) =: \mathcal{T}^p_{poly}(M).$$

The bracket induced on $\mathcal{T}_{poly}(M)$ is -(up to a sign $[T_1, T_2]_{\mathcal{T}} := -[T_2, T_1]_S$) - the **Schouten-Nijenhuis bracket** defined by extending the usual Lie bracket of vector fields

$$[X_1 \wedge \ldots \wedge X_k, Y_1 \wedge \ldots \wedge Y_l]_S = \sum_{r=1}^k \sum_{s=1}^l (-1)^{r+s} [X_r, X_s] X_1 \wedge \ldots \widehat{X_r} \wedge \ldots \wedge X_k \wedge Y_1 \wedge \ldots \widehat{Y_s} \wedge \ldots \wedge Y_l.$$

$(\mathcal{T}_{poly}(M), [\,,\,]_{\mathcal{T}}, 0)$ is a DGLA.

A $P \in \nu\mathcal{T}^1_{poly}(M)[[\nu]]$ defines a formal Poisson structure on M iff $d_{\mathcal{T}} P - \frac{1}{2}[P, P]'_S = 0$.

Poisson deformations and the DGLA of polyvectorfields

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Maps between the DGLA's

The natural map $U_1\colon \mathcal{T}^i_{poly}(M) \longrightarrow \mathcal{D}^i_{poly}(M)$

$$U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)}), \tag{4}$$

intertwines the differential and induces the identity in cohomology, but is not a DGLA morphism.

A DGLA morphism from $(\mathcal{T}_{poly}(M), [\,,\,]_{\mathcal{T}}, 0)$ to $\big(\mathcal{D}_{poly}(M), [\,,\,]_G, d_{\mathcal{D}}\big)$, inducing the identity in cohomology, would give a correspondence between a formal Poisson tensor on M and a formal differential star product on M and a bijection between equivalence classes.

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L_∞ algebras

Let $W = \oplus_{j\in\mathbb{Z}} W^j$ be a $\mathbb{Z}$-graded vector space; $V = W[1]$ is the shifted graded vector space.

The graded symmetric bialgebra of V, denoted $\mathcal{S}V$, is the quotient of the free algebra $\mathcal{T}V$ by the two-sided ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$ for any homog. elements x, y in V.

Δ_{sh} is induced by $\Delta_{sh} : \mathcal{T}V \to \mathcal{T}V \otimes \mathcal{T}V$ which is the morphism of assoc. algebras so that $\Delta_{sh}(x) = 1 \otimes x + x \otimes 1$.

A L_∞-structure on W is defined to be a graded coderivation $\mathcal{Q}$ of $\mathcal{S}(W[1])$ of degree 1 satisfying $\mathcal{Q}^2 = 0$ and $\mathcal{Q}(\mathbf{1}_{\mathcal{S}W[1]}) = 0$.

Such a $\mathcal{Q}$ is determined by $Q := pr_{W[1]} \circ \mathcal{Q} : \mathcal{S}(W[1]) \to W[1]$ via $\mathcal{Q} = \mu_{sh} \circ Q \otimes \mathrm{Id} \circ \Delta_{sh}$ and we write $\mathcal{Q} = \overline{Q}$.

The pair $(W, \mathcal{Q})$ is called an L_∞-**algebra** .

Ex: $(\mathfrak{g}, [\ ,\ ], d)$ a DGLA $\Rightarrow \left(\mathfrak{g}, \mathcal{Q} = \overline{d[1] + [\ ,\ ][1]}\right)$ (with $\mathcal{Q}$ defined on $\mathcal{S}(\mathfrak{g}[1])$).

For $\phi : V^{\otimes k} \to W^{\otimes \ell}$, $\phi[j] : V[j]^{\otimes k} \to W[j]^{\otimes \ell}$ via $\phi[j] := (s^{\otimes \ell})^{-j} \circ \phi \circ (s^{\otimes k})^j$ where $s : V \to V[-1]$ is the identity.

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Quasi-isomorphisms and Formality

A L_∞**-morphism** from a L_∞-algebra $(W, \mathcal{Q})$ to a L_∞-algebra $(W', \mathcal{Q}')$ is a morphism of graded con. coalgebras $\Phi : \mathcal{S}(W[1]) \to \mathcal{S}(W'[1])$, intertwining differentials

$$\Phi \circ \mathcal{Q} = \mathcal{Q}' \circ \Phi.$$

Such a morphism is determined by $\varphi := pr_{W'[1]} \circ \Phi : \mathcal{S}(W[1]) \to W'[1]$ with $\varphi(1) = 0$ via $\Phi = e^{*\varphi}$ with $A * B = \mu \circ A \otimes B \circ \Delta$ for $A, B \in \mathrm{Hom}(\mathcal{S}(W[1]), \mathcal{S}(W'[1]))$

Φ is a **quasi-isomorphism** if $\Phi_1 = \Phi|_{W[1]} = \varphi_1 : W[1] \to W'[1]$ induces an iso. in cohomology.

A **formality for a DGLA** $(\mathfrak{g}, [\ ,\ ], d)$ is a quasi-isomorphism from the L_∞-algebra corresponding to $(H, [\ ,\ ]_H, 0)$ (the cohomology of $\mathfrak{g}$ with respect to d), to the L_∞-algebra corresponding to $(\mathfrak{g}, [\ ,\ ], d)$ i.e.

$$\Phi : \mathcal{S}(\mathfrak{H}[1]) \to \mathcal{S}(\mathfrak{g}[1]) \text{ such that } \Phi \circ \overline{[\ ,\ ]_H[1]} = \overline{(d[1] + [\ ,\ ][1])} \circ \Phi.$$

A quasi-isomorphism yields isomorphic moduli spaces of deformations.

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Kontsevich's formality for $\mathbb{R}^d$

Kontsevich gave an explicit formula for the Taylor coefficients of a formality for $\mathbb{R}^d$, i.e. the Taylor coefficients F_n of an L_∞–morphism between the two L_∞-algebras

$$F : (\mathcal{T}_{poly}(\mathbb{R}^d), \mathcal{Q}) \to (\mathcal{D}_{poly}(\mathbb{R}^d), \mathcal{Q}')$$

corresponding to the DGLA $(\mathcal{T}_{poly}(\mathbb{R}^d)\ ,\ [\ \ ,\ ]_{\mathcal{T}},\ d_{\mathcal{T}} = 0)$ and to the DGLA $(\mathcal{D}_{poly}(\mathbb{R}^d)\ ,\ [\ ,\ ]_G\ ,\ d_{\mathcal{D}})$ with the first coefficient

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given by $F_1 = U_1\ :\ U_1(X_0 \wedge \ldots \wedge X_n)(f_0, \ldots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \epsilon(\sigma)\, X_0(f_{\sigma(0)}) \cdots X_n(f_{\sigma(n)})$. The formula is

$$F_n = \sum_{m\geq 0} \sum_{\vec{\Gamma}\in G_{n,m}} \mathcal{W}_{\vec{\Gamma}} B_{\vec{\Gamma}}$$

where $G_{n,m}$ is a set of oriented admissible graphs; $B_{\vec{\Gamma}}$ asoociates a m–differential operator to an n–tuple of multivectorfields; and $\mathcal{W}_{\vec{\Gamma}}$ is the integral of a form $\omega_{\vec{\Gamma}}$ over the compactification of a configuration space $C^+_{\{p_1,\ldots,p_n\}\{q_1,\ldots,q_m\}}$.

An explicit globalisation on any manifold has been built by Cattaneo, Felder and Tomassini.

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Classification of star products.

Given a Poisson tensor $P_{(\nu)} = \sum_{k\geq 1} \nu^k P_k$, then $\star_{P_{(\nu)}} := \mu + \sum_{k\geq 1} F_k(P_{(\nu)}, \cdots, P_{(\nu)})$ is a star product on (M, P_1) and any $\star$ product is equivalent to such a one. Equivalence classes of star products are in bijection with equivalence classes of Poisson deformations.

Hermitian star products and $*$-algebras over ordered rings

To study representations of the deformed algebras, parts of the algebraic theory of states and representations which exist for C^*-algebras have been extended by Bordemann, Bursztyn and Waldmann to the framework of $*$-algebras over ordered rings

A C^*-algebra is a Banach algebra over $\mathbb{C}$ with a $*$ involution (i.e. an involutive semilinear antiautomorphism) such that $\|a\| = \|a^*\|$ and $\|aa^*\| = \|a\|^2$ for each a .
If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and if $0 \neq \psi \in \mathcal{H}$,

$$\omega : \mathcal{A} \to \mathbb{C} : A \mapsto \omega(A) := \frac{< \psi, A\psi >}{< \psi, \psi >}$$

is a linear functional which is positive in the sense that $\omega(A^*A) \geq 0$. It is defined by the ray ψ generates. This lead to define a state in the theory of C^* algebras as a positive linear functional.

An associative commutative unital ring R is said to be **ordered** with positive elements P if the product and the sum of two elements in P are in P, and if R is the disjoint union $R = P \cup \{0\} \cup -P$. Examples are given by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}[[\lambda]]$; in the case of $\mathbb{R}[[\lambda]]$, a series $a = \sum_{r=r_0}^{\infty} a_r \lambda^r$ is positive if its lowest order non vanishing term is positive ($a_{r_0} > 0$).

Let R be an ordered ring and $C = R(i)$ be the ring extension by a square root i of -1 (for deformation quantization, $C = \mathbb{C}[[\lambda]]$ for $R = \mathbb{R}[[\lambda]]$ with $\nu = i\lambda$).

An associative algebra $\mathcal{A}$ over C is called a $*$**-algebra** if it has an involutive antilinear antiautomorphism $* : \mathcal{A} \to \mathcal{A}$ called the $*$-involution;for instance $(C^\infty(M)[[\nu = i\lambda]], \star)$ with a Hermitian star product and conjugaison is a $*$-algebra over $\mathbb{C}[[\lambda]]$.

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An associative algebra $\mathcal{A}$ over C is called a *-**algebra** if it has an involutive antilinear antiautomorphism $^* : \mathcal{A} \to \mathcal{A}$ called the *-involution; for instance $(C^\infty(M)[[\nu = i\lambda]], \star)$ with a Hermitian star product and conjugaison is a *-algebra over $\mathbb{C}[[\lambda]]$.

Hermitian star products and $*$-algebras over ordered rings

To study representations of the deformed algebras, parts of the algebraic theory of states and representations which exist for C^*-algebras have been extended by Bordemann, Bursztyn and Waldmann to the framework of $*$-algebras over ordered rings

A C^*-algebra is a Banach algebra over $\mathbb{C}$ with a $*$ involution (i.e. an involutive semilinear antiautomorphism) such that $\|a\| = \|a^*\|$ and $\|aa^*\| = \|a\|^2$ for each a .
If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and if $0 \neq \psi \in \mathcal{H}$,

$$\omega : \mathcal{A} \to \mathbb{C} : A \mapsto \omega(A) := \frac{< \psi, A\psi >}{< \psi, \psi >}$$

is a linear functional which is positive in the sense that $\omega(A^*A) \geq 0$. It is defined by the ray ψ generates. This lead to define a state in the theory of C^* algebras as a positive linear functional.

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An associative algebra $\mathcal{A}$ over C is called a ***-algebra** if it has an involutive antilinear antiautomorphism $^* : \mathcal{A} \to \mathcal{A}$ called the *-involution;for instance $(C^\infty(M)[[\nu = i\lambda]], \star)$ with a Hermitian star product and conjugaison is a *-algebra over $\mathbb{C}[[\lambda]]$.

States for Hermitian star products

A linear functional $\omega : \mathcal{A} \to C$ over a *-algebra over C is called **positive** if

$$\omega(A^*A) \geq 0 \qquad \text{for any } A \in \mathcal{A}.$$

A **state** for a *-algebra $\mathcal{A}$ with unit over C is a positive linear functional so that $\omega(1) = 1$.
The positive linear functionals on $C^\infty(M)$ are the compactly supported Borel measures. The δ-functional on $\mathbb{R}^{2n}$ is not positive with respect to the Moyal star product : if $H := \frac{1}{2m}p^2 + kq^2$, $(H \star_M H)(0,0) = \frac{k\nu^2}{2m} = \frac{-k\lambda^2}{2m} < 0$. Bursztyn and Waldmann proved that for a Hermitian star product, any classical state ω_0 on $C^\infty(M)$ can be deformed into a state for the deformed algebra, $\omega = \sum_{r=0}^\infty \lambda^r \omega_r$.

Given a positive functional ω over the *-algebra $\mathcal{A}$, one can extend the GNS construction of an associated representation of the algebra: the **Gel'fand ideal of** ω is $\mathcal{J}_\omega = \left\{ a \in \mathcal{A} \,\middle|\, \omega(a^*a) = 0 \right\}$ and on obtains the **GNS- representation** of the algebra $\mathcal{A}$ by left multiplication on the space $\mathcal{H}_\omega = \mathcal{A}/\mathcal{J}_\omega$ with the pre Hilbert space structure defined via $\langle [a], [b] \rangle = \omega(a^*b)$ where $[a] = a + \mathcal{J}_\omega$ denotes the class in $\mathcal{A}/\mathcal{J}_\omega$ of $a \in \mathcal{A}$.

In that setting, Bursztyn and Waldmann introduced a notion of strong Morita equivalence (yielding equivalence of $*$-representations) and the complete classification of star products up to Morita equivalence was given, first on a symplectic and later in collaboration with Dolgushev on a general Poisson manifold .

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