Deformation Quantization and Symmetries
Episode 3

Programme

Introduction to the concept of deformation quantization (existence, classification and representation results for formal star products).

Notion of formal star products with symmetries;

Group Action in a classical setting

A Lie group G acts by **Poisson diffeomorphisms** on (M, P) iff

$$\{g^*u, g^*v\} = g^*(\{u, v\}) \qquad \forall u, v \in C^\infty(M), \forall g \in G,$$

or, equivalently, if and only if $g_*P = P$ for all $g \in G$.
In the symplectic case, this is equivalent to $g^*\omega = \omega$ for all $g \in G$.
Then G is a symmetry group for our classical system.

Any X in the Lie algebra $\mathfrak{g}$ of G gives rise to a fundamental vector field X^{*M}

$$X^{*M}_p = \frac{d}{dt}_{vert0} \exp -tX \cdot p;$$

then $[X^{*M}, Y^{*M}] = [X, Y]^{*M}$ and we have an infinitesimal Poisson action of $\mathfrak{g}$

$$\mathcal{L}_{X^{*M}}\{u, v\} = \{\mathcal{L}_{X^{*M}} u, v\} + \{u, \mathcal{L}_{X^{*M}} v\} \tag{1}$$

or equivalently $\mathcal{L}_{X^{*M}} P = 0$ or, in the symplectic case $\mathcal{L}_{X^{*M}}\omega = 0$ which says that $\iota(X^{*M})\omega$ is a closed 1-form on M.

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Group action in the deformation quantization setting

The action of a Lie group on the classical Hilbert space framework of quantum mechanics is described by a unitary representation of the group on the Hilbert space; such a representation acts by conjugaison on the set of selfadjoint operators on that space and yields an automorphism of the algebra of quantum observables.

In the setting of deformation quantization, the classical action of a group G on a Poisson manifold extends to the algebra of observables $C^\infty(M)[[\nu]]$ and one can define different notions of invariance of the deformation quantization under the action of a Lie group.

Assume (M, P) is a Poisson manifold and G is a Lie group acting on M. Let $(\mathbb{C}^\infty(M)[[\nu]], \star)$ be a deformation quantization of (M, P). The star product is said to be **geometrically invariant** if,

$$g^*(u \star v) = g^*u \star g^*v \qquad \forall g \in G, \forall u, v \in C^\infty(M).$$

This clearly implies that $g^*(\{u, v\}) = \{g^*u, g^*v\}$ so G acts by Poisson diffeomorphisms. X^{*M} is then a derivation of the star product $X^{*M}(u \star v) = (X^{*M}u) \star v + u \star (X^{*M}v)$.

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Symmetries of star products

More generally, symmetries in quantum theories are automorphisms of the algebra of observables.
A symmetry σ of a star product $\star$ is an automorphism of the $\mathbb{C}[[\nu]]$-algebra $(C^\infty(M)[[\nu]], \star)$

$$\sigma(u \star v) = \sigma(u) \star \sigma(v), \qquad \sigma(1) = 1,$$

where σ is a formal series of linear maps.
One can show that $\sigma(u) = T(u \circ \tau)$ where τ is a Poisson diffeomorphism of (M, P) and $T = \mathrm{Id} + \sum_{r \geq 1} \nu^r T_r$ a formal series of differential maps.

A Lie group G acts as **symmetries of** $(C^\infty(M)[[\nu]], \star)$ if there is a homomorphism

$$\sigma : G \to \mathrm{Aut}(M, \star).$$

In that case, $\sigma(g)u = T(g)(\tau(g)^* u)$ and τ defines a Poisson action of G on (M, P).

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Invariant star products and invariant connections

Two G-invariant star products are called **G-equivariantly equivalent** if there is an equivalence between them which commutes with the action of G.

Existence and classification of invariant star products on a Poisson manifold is known, provided there exists an invariant connection on the manifold.

Fedosov's construction in the symplectic case builds a star product $\star_{\nabla,0}$, from a symplectic connection ∇. If ∇ is invariant under the action of G, $\star_{\nabla,0}$ is invariant.

More generally, any diffeomorphism ϕ of (M,ω) is a symmetry of the Fedosov star product $\star_{\nabla,\tilde{\Omega}}$ iff it preserves the symplectic 2-form ω, the connection ∇ and the series of closed 2-forms $\tilde{\Omega}$.

Reciprocally [Rawnsley-G], a natural star product on a symplectic manifold determines in a unique way a symplectic connection. Hence, when G acts on (M,ω) and leaves a natural $\star$ product invariant, there is a unique symplectic connection which is invariant under G.

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Classification of invariant star products

Suppose $\star$ is G-invariant on (M,ω) and assume there exists a G-invariant symplectic connection ∇. Then, there exists a series of G-invariant closed 2-form $\Omega \in Z^2(M;\mathbb{R})^{G-inv}[[\nu]]$ such that $\star$ is G-equivalent to the Fedosov star product constructed from ∇ and Ω. Furthermore $*_{\nabla,\Omega}$ and $*_{\nabla,\Omega'}$ are G- equivalent if and only if $\Omega - \Omega'$ is the boundary of a series of G-invariant 1-forms on M.

Hence [Bertelson, Bieliavsky, G.],there is a bijection between the G-equivalence classes of G-invariant $*$-products on (M,ω) and the space of formal series of elements in the second space of invariant cohomology of M, $H^2(M,\mathbb{R})^{G-inv}[[\nu]]$.

Using Dolgushev's construction of a formality starting from a connection, one has a similar result in the Poisson setting : If there exists an invariant connection, there is a bijection between the G-equivalence classes of G-invariant $*$-products on (M,P) and the G-equivariant equivalence classes of G-invariant Poisson deformations of P.

Let us mention that there exist symplectic manifolds which are G-homogeneous but do not admit any G-invariant symplectic connection. A first example was given by Arnal: the orbit of a filiform nilpotent Lie group in the dual of its algebra.

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Invariant star products on Lie groups and Drinfeld twists

The class of manifolds with a simply transitive action are Lie groups with the action given by left multiplication; one is interested in left invariant $\star$-products on Lie groups.

Since left invariant differential operators on a Lie group G are identified with elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, left invariant bidifferential operators can be viewed as elements of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and a left invariant $\star$-product on a Lie group G is given by an element

$$F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\nu]].$$

such that

- $(\Delta \otimes \mathrm{Id})(F) \circ (F \otimes 1) = (\mathrm{Id} \otimes \Delta)(F) \circ (1 \otimes F)$ where $\circ$ denotes the product in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is the usual coproduct ($\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is the algebra morphism such that $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$.) , both extended $\mathbb{C}[[\nu]]$-linearly ; this expresses the associativity;
- $(\epsilon \otimes \mathrm{Id})F = 1 = (\mathrm{Id} \otimes \epsilon)F$, where $\epsilon : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ is the counit; this expresses that $1 \star u = u \star 1 = u$;
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Since left invariant differential operators on a Lie group G are identified with elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, left invariant bidifferential operators can be viewed as elements of $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and a left invariant $\star$-product on a Lie group G is given by an element

$$F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))\,[[\nu]].$$

such that

- $(\Delta \otimes \mathrm{Id})(F) \circ (F \otimes 1) = (\mathrm{Id} \otimes \Delta)(F) \circ (1 \otimes F)$ where $\circ$ denotes the product in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ and $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is the usual coproduct ($\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is the algebra morphism such that $\Delta(x) = 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$.) , both extended $\mathbb{C}[[\nu]]$-linearly ; this expresses the associativity;
- $(\epsilon \otimes \mathrm{Id})F = 1 = (\mathrm{Id} \otimes \epsilon)F$, where $\epsilon : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ is the counit; this expresses that $1 \star u = u \star 1 = u$;
- $F = 1 \otimes 1 + O(\nu)$, which expresses that the zeroth order term is the usual product of functions.

Such an element is called a **formal Drinfeld twist**.

r-matrices and Drinfeld twists.

The skewsymmetric part of the first order term, which is automatically in $\mathfrak{g} \otimes \mathfrak{g}$ corresponds to a left invariant Poisson structure on G and is what is called a **classical r-matrix.**

Drinfeld has proven in 83 that any classical r-matrix arises as the first term of a Drinfeld twist (see Halbout about formality of bialgebras , or Esposito, Schnitzer and Waldmann in 2017 about a universal construction).

An invariant equivalence is given by an element $S \in \mathcal{U}(\mathfrak{g})[[\nu]]$ of the form $S = 1 + O(\nu)$ and the equivalent $\star$-product is defined by the new Drinfeld twist given by

$$F' = \Delta(S^{-1})F(S \otimes S).$$

An analogous algebraic construction on a homogeneous space $M = G/H$ was given by Alekseev and Lachowska in 2005 ; invariant bidifferential operators on G/H are viewed as elements of $\left((\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h})^{\otimes 2}\right)^H$; a star product is given in terms of a series $B \in \left((\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h})^{\otimes 2}\right)^H [[\nu]]$ and associativity writes again as $((\Delta \otimes \mathrm{Id})B)(B \otimes 1) = ((\mathrm{Id} \otimes \Delta)B)(1 \otimes B)$ where both sides define uniquely invariant tri-differential operators on G/H.

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Universal Deformation Formulas

Given a left invariant star product on a Lie group, hence a formal Drinfeld twist $F \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\nu]]$ on its Lie algebra $\mathfrak{g}$, one can deform any associative algebra (A, μ_A) acted upon by $\mathfrak{g}$ through derivations.

This process is called a **universal deformation formula** and is defined as follows:

$$a \star_F b := \mu_A (F \bullet (a \otimes b))$$

where $\bullet$ denotes the action of $\mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g})[[\nu]]$ on $A \times A[[\nu]]$ which is the extension of the action of $\mathfrak{g}$ on A to an action of $\mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g})$ on $A \times A$ extended $\mathbb{C}[[\nu]]$-linearly. The properties of a twist ensure that $\star_F$ is an associative deformation of μ_A. (Giaquinto and Zhang in 1998, by Bieliavsky and Gayral in a non formal setting, Esposito, Schnitzer and Waldmann).

Programme

Introduction to the concept of deformation quantization (existence, classification and representation results for formal star products).

Notion of formal star products with symmetries; one has a Lie group action (or a Lie algebra action) compatible with the classical Poisson structure, and one wants to consider star products such that the Lie group acts by automorphisms (or the Lie algebra acts by derivations). We recall in particular the link between left invariant star products on Lie groups and Drinfeld twists, and the notion of universal deformation formulas.

Quantum moment map : Classically, symmetries are particularly interesting when they are implemented by a moment map and we give indications to build a corresponding quantum version.

Hamiltonian Action in a classical setting

Of particular importance in physics is the case where the action is implemented by a moment map. The action is called **(almost) Hamiltonian** when each fundamental vector field is Hamiltonian, i.e. when for each $X \in \mathfrak{g}$ there exists a function f_X on M such that

$$X^{*M} u = \{f_X, u\} \qquad \forall u \in C^\infty(M).$$

In the symplectic case this amounts to say that $\iota(X^{*M})\omega = df_X$.

When the Hamiltonian governing the dynamics on (M, P) is invariant under the action of G, any of those functions f_X is a constant of the motion.

A further assumption is to ask that the action possesses a G **equivariant moment map** J:

$$J : M \to \mathfrak{g}^* \quad \text{s.t.} \quad X^{*M} u = \{< J, X >, u\} \quad \forall u \in C^\infty(M)$$

$< J, X >: M \to \mathbb{R} : p \mapsto < J(p), X >$, $< ., . >$ denoting the pairing between $\mathfrak{g}$ and its dual.
Equivariance means that the Hamiltonian functions $f_X := < J, X >$ satisfy $f_X(g \cdot p) = f_{Adg^{-1}X}(p)$ and thus

$$\{ f_X, f_Y \} = f_{[X,Y]} \qquad \forall X, Y \in \mathfrak{g}.$$

An action so that each fundamental vector field is Hamiltonian and so that one can choose $X \mapsto f_X$ to be a homomorphism of Lie algebras is also called a **(strongly) Hamiltonian action**. If G is connected, it is equivalent to the existence of a G equivariant moment map.

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Hamiltonian action in the deformation quantization setting

An action of the Lie algebra $\mathfrak{g}$ on the deformed algebra, $(C^\infty(M)[\nu]]], \star)$, is a homomorphism $D : \mathfrak{g} \to \mathrm{Der}(M, \star)$ into the space of derivations of the star product.

A derivation D is **essentially inner** or **Hamiltonian** if $D = \frac{1}{\nu}\,\mathrm{ad}_\star\, u$ for some $u \in C^\infty(M)[[\nu]]$.

We call an action of a Lie algebra (or of a Lie group) on a deformed algebra **almost $\star$-Hamiltonian** if each $D(X)$, for any $X \in \mathfrak{g}$, is essentially inner, and we call (quantum) Hamiltonian a linear choice of functions u_X satisfying

$$D(X) = \frac{1}{\nu}\,\mathrm{ad}_\star\, u_X, \qquad X \in \mathfrak{g}.$$

The action is **$\star$-Hamiltonian** if u_X can be chosen to make the map

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a homomorphism of Lie algebras (i.e. $\frac{1}{\nu}(u_X \star u_Y - \star u_Y \star u_X) = u_{[X,Y]} \quad \forall X, Y \in \mathfrak{g}$).

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Quantum moment maps

When $\star$ is invariant under the action of G on (M,P) and the corresponding action of the Lie algebra $\mathfrak{g}$ (given by $D(X) = X^{*M}$) is $\star$-Hamiltonian, a map $\mathfrak{g} \to C^\infty(M)[[\nu]]$ s.t.

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is called **a quantum moment map** [Xu].

For a (strongly) Hamiltonian action of Lie group G on (M,P), with $f : \mathfrak{g} \to \mathbb{C}^\infty(M)$ describing the classical moment map (i.e. $X^{*M}u = \{f_X, u\}$),

- a star product is said to be **covariant** under G if

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- and is called **strongly invariant** if it is geometrically invariant and

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  In that case, f is a quantum moment map.

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Invariance of a Fedosov star product $\star_{\nabla,\tilde{\Omega}}$

Any diffeomorphism ϕ of (M,ω) is a symmetry of $\star_{\nabla,\tilde{\Omega}}$ iff it preserves the symplectic 2-form ω, the connection ∇ and the series of closed 2-forms $\tilde{\Omega}$.

X is a derivation of $\star_{\nabla,\tilde{\Omega}}$ iff $\mathcal{L}_X\omega = 0$, $\mathcal{L}_X\tilde{\Omega} = 0$, and $\mathcal{L}_X\nabla = 0$.

X is an inner derivation of $\star_{\nabla,\Omega}$ iff $\mathcal{L}_X\nabla = 0$ and $\exists\lambda_X \in C^\infty(M)[[\nu]]$ such that

$$i(X)\omega - i(X)\tilde{\Omega} = d\lambda_X.$$

In this case $X(u) = \frac{1}{\nu}(\mathrm{ad}_\star \lambda_X)(u)$ [G. -Rawnsley, Bahns-Neumaier, Kravchenko].

A $\mathfrak{g}$ -invariant Fedosov star product for (M,ω) is obtained from a $\mathfrak{g}$ invariant connexion and a $\mathfrak{g}$ invariant series of closed 2-forms Ω. It admits a quantum moment map if and only if there is a linear map $J : \mathfrak{g} \to C^\infty(M)[[\nu]]$ such that

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Link with equivariant cohomology

If one considers a pair $(\star, J)$ of an $\mathfrak{g}$-invariant star-product and a quantum moment map, there is a natural notion of equivalence : two such pairs $(\star, J)$ and $(\star', J')$, are "equivariantly" equivalent if there is a $\mathfrak{g}$-invariant equivalence T between $\star$ and $\star'$ such that $J' = TJ$.

On any symplectic manifold (M, ω) equipped with a $\mathfrak{g}$-strongly hamiltonian action with moment map J and a $\mathfrak{g}$-invariant connection, the "equivariant" equivalence classes of such pairs are parametrized by series of second equivariant cohomology classes $(\frac{[\omega - J]}{\nu} + H^2_{\mathfrak{g}}(M)[[\nu]])$ [Reichert-Waldmann, 2017].

Let M be a manifold, $\mathfrak{g}$ be a Lia algebra and $\rho : \mathfrak{g} \to \chi(M)$ a Lie algebra action of $\mathfrak{g}$ on M. The complex of $\mathfrak{g}$-equivariant forms $\Omega_{\mathfrak{g}}(M)$ is defined as

$$\Omega_{\mathfrak{g}}(M) := \left(\oplus_{2i+j=k}[S^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^{\mathfrak{g}}\,,\ , d_{\mathfrak{g}} := d + \iota_\bullet\right)$$

where invariants are taken with respect to $X \cdot (p \otimes \alpha) = (\mathrm{ad}^*_X p \otimes \alpha + p \otimes (\mathcal{L}_{\rho(X)}\alpha)$ and where $\iota_\bullet$ denotes the insertion of a vector field in the first component of the differential form part : $(\iota_\bullet(p \otimes \alpha))(X) = \iota(\rho(X))((p \otimes \alpha)(X))$ when $p \otimes \alpha$ is viewed as a polynomial on $\mathfrak{g}$ with values in the space $\Omega(M)$.

Link with equivariant cohomology

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Derivations of a Kontsevich star product $\star_P$

Given a Poisson structure P and a vector field X so that $\mathcal{L}_X P = 0$, then

$$A_X = X + \sum_{k\geq 1} \nu^k F_{k+1}(X, P, \cdots, P)$$

is automatically a derivation of the $\star$-product $\star_P = \mu + \sum_{k\geq 1} \nu^k F(X, P, \cdots, P)$.

If X, Y are two vector fileds M preserving P then

$$[A_X, A_Y] = A_{[X,Y]} + \sum_{k\geq 1} \nu^k F_{k+2}(X, Y, P, \cdots, P).$$

Esposito, de Kleijn and Schnitzer have recently proven an equivariant version of formality of multidifferential operators for a proper Lie group action; this allows to obtain a quantum moment map from a classical moment map with respect to a G-invariant Poisson structure and generalizes the theorem cited above from the symplectic setting to the Poisson setting.

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Invariant star products on coadjoint orbits

A natural class of symplectic manifolds on which there is a strongly hamiltonian action of a Lie group is the class of coadjoint orbits in Lie groups in the dual of their algebras.

Their interest comes from the Kirillov-Souriau-Kostant orbit method in representation theory which associates certains irreducible unitary representations of a given Lie group to some of its coadjoint orbits.

Much work has been devoted to the construction of interesting star-products on these orbits.

Remark that those orbits do not always possess an invariant connection so one can not hope to get in all cases an invariant star-product!