Deformation Quantization and Symmetries
Episode 4

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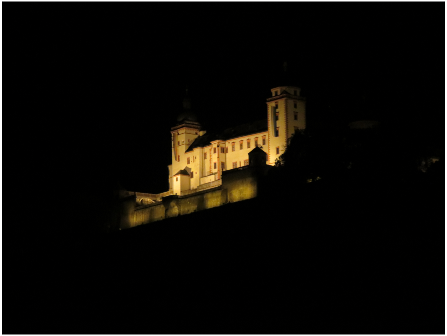

Programme

Introduction to the concept of deformation quantization (existence, classification and representation results for formal star products).

Notion of formal star products with symmetries; one has a Lie group action (or a Lie algebra action) compatible with the classical Poisson structure, and one wants to consider star products such that the Lie group acts by automorphisms (or the Lie algebra acts by derivations). We recall in particular the link between left invariant star products on Lie groups and Drinfeld twists, and the notion of universal deformation formulas.

Quantum moment map : Classically, symmetries are particularly interesting when they are implemented by a moment map. We give indications to build a corresponding quantum version. Concerning links between representation theory and the quantization of an orbit of a group in the dual of its Lie algebra, we recall how some star products yield an adapted Fourier transform.

The orbit method in representation theory

The aim is to describe the unitary dual $\hat{G}$ of the Lie group G ($\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G) from the coadjoint orbits of G.

If π is a unitary representation of G on $\mathcal{H}$ and ϕ an integrable function on G,

$$\pi(\phi) := \int_G \phi(g)\pi(g)dg \qquad \text{with } dg \text{ a fixed Haar measure on } G$$

is a bounded operator.

For a class of unimodular Lie group G, there exists a measure on $\hat{G}$ called the Plancherel measure such that

$$\int_G |\phi|^2 dg = \int_{\hat{G}} \operatorname{tr}\left((\pi(\phi)(\pi(\phi))^*\right) dm(\pi) \qquad \forall \phi \in L^2(G) \cap L^1(G).$$

For a large subclass of representations, the operator $(\pi(\phi))$ for ϕ smooth with compact support is tracable, and the map the map $\phi \to \operatorname{Tr}(\pi(\phi))$ is called the character distribution of the representation.

An orbit O is said to be associated with the representation π if there is a so called Fourier transform $\mathcal{F} : C_c^\infty(G) \to Dist(\mathfrak{g}^*)$ such that

$$\operatorname{Tr}(\pi(\phi)) = \int_O F(\phi) d\mu_O$$

where $d\mu_O$ is the symplectic volume form of the orbit.

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Deformation quantization and adapted Fourier transform

In deformation quantization, the idea is to build an adapted Fourier transform $\mathcal{E} : C^\infty_c(G) \to Dist(\mathfrak{g}^*)$ of the form

$$(\mathcal{E}(\phi))(\xi) = \int_G \phi(g)(E_O(g))(\xi)dg \qquad \text{for } \xi \in O$$

where $E_O(g)$ is obtained from a covariant star product on O. It should again yield

$$\mathrm{Tr}(\pi(\phi)) = \int_O \mathcal{E}(\phi) d\mu_O$$

where $d\mu_O$ is the symplectic volume form of the orbit.

Some invariant star products on coadjoint orbits

For a nilpotent Lie group, Arnal and Cortet have built a covariant star product using Moyal star product in good adapted coordinates. They showed that a covariant star product gives rise to a representation of the group into the automorphisms of the star product. One can define the star exponential of the elements in the Lie algebras, and this gives a construction of adapted Fourier transforms. They extended their construction to orbits of exponential solvable groups.

For a compact group G, a star product was obtained by asymptotic expansion of the translation at the level of Berezin's symbols of the composition of operators acting on the finite Hilbert spaces of sections of powers of a line bundle built on the Kähler manifold G/T [Cahen-G-Rawnsley].

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Berezin's symbols

Let $(L \xrightarrow{\pi} M, \nabla, h)$ be a quantization bundle over the compact Kähler manifold (M, ω, J) (i.e., L is a holomorphic line bundle with connection ∇ admitting an invariant hermitian structure h, such that the curvature is $\mathrm{curv}(\nabla) = -2i\pi\omega$).
Let $\mathcal{H}$ be the Hilbert space of holomorphic sections of L.
Since evaluation at a point is a continuous linear functional on $\mathcal{H}$, let, for any $q \in L_0$ let e_q be the so-called **coherent state** defined by

$$s(\pi(q)) = < s, e_q > q \quad \text{for any} \quad s \in \mathcal{H};$$

then $e_{cq} = \overline{c}^{-1} e_q$ for any $0 \neq c \in \mathbb{C}$, and let ϵ be the **characteristic function** on M defined by $\epsilon(x) = \|q\|^2 \|e_q\|^2$, with $q \in L_0$ so that $\pi(q) = x$.
Any linear operator $\mathbf{A}$ on $\mathcal{H}$ has a **Berezin's symbol**

$$\hat{A}(x) := \frac{< \mathbf{A} e_q, e_q >}{\|e_q\|^2} \quad q \in L_0, \pi(q) = x \in M \tag{1}$$

which is a real analytic function on M. The operator can be recovered from its symbol:

$$(\mathbf{A}s)(x) := \int_M h_y(s(y), e_q(y)) \hat{A}(x, y) \frac{\omega^n(y)}{n!} q \qquad s \in \mathcal{H}, q, q' \in L_0, \pi(q) = x, \pi(q') = y,$$

where $\hat{A}(x, y) := \frac{< \mathbf{A} e_{q'}, e_q >}{< e_{q'}, e_q >}$ is the analytic continuation of the symbol, holomophic in x and antiholomorphic in y, defined on the open dense set of $M \times M$ consisting of points (x, y) such that $< e_{q'}, e_q > \neq 0$. Denote by $\hat{E}(L)$ the space of these symbols.

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Star products from Berezin's symbols

For any positive integer k, $(L^k = \otimes^k L, \nabla^{(k)}, h^{(k)})$ is a quantization bundle for $(M, k\omega, J)$.
If $\mathcal{H}^k$ is the Hilbert space of holomorphic sections of L^k, we denote by $\hat{E}(L^k)$ the space of symbols of linear operators on $\mathcal{H}^k$.

If, for every k, the characteristic function $\epsilon^{(k)}$ on M is constant (which is true in a homogeneous case), one says that the quantization is **regular**.
In that case, the space $\hat{E}(L^l)$ is contained in the space $\hat{E}(L^k)$ for any $k \geq l$.

Furthermore $\mathcal{C}_L := \cup_{l=1}^{\infty} \hat{E}(L^l)$ is a dense subspace of the space of continuous functions on M.
Any function f in $\mathcal{C}_L$ belongs to a particular $\hat{E}(L^l)$ and is thus the symbol of an operator $\mathbf{A}_f^{(k)}$ acting on $\mathcal{H}^k$ for $k \geq l$. One has thus constructed, for a given f, a family of quantum operators parametrized by an integer k. From the point of view of deformation theory, one has constructed a family of associative products $*_k$ on $\hat{E}(L^l)$, with values in $\mathcal{C}_L$, parametrized by an integer k :

$$f *_k g = \widehat{\mathbf{A}_f^{(k)} \mathbf{A}_g^{(k)}} \qquad f, g \in \hat{E}(L^l);\ k \geq l. \tag{2}$$

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Quantum reduction : reduction is a construction in classical mechanics with symmetries which allows to reduce the dimension of the manifold; we describe one of the various quantum analogues which have been considered in the framework of formal deformation quantization.

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Reduction in Poisson geometry

Reduction is an important classical tool to "reduce the number of variables", (start from a "big" Poisson manifold (M, P) and construct a smaller one (M_{red}, P_{red})).
Consider an embedded coisotropic submanifold in the Poisson manifold,

$$\iota : C \hookrightarrow M.$$

A submanifold of a Poisson manifold is called coisotropic iff the vanishing ideal

$$\mathcal{J}_C = \{f \in C^\infty(M) \mid \iota^* f = 0\} = \ker \iota^*.$$

is closed under Poisson bracket. This is equivalent to say that $P^\sharp(N^*C) \subset TC$ where $N^*C(x) = \{\, \alpha_x \in T_x^*M \mid \alpha_x(X) = 0 \,\forall X \in T_xC \,\}$ and where $P^\sharp : T^*M \to TM$ is defined by $\beta(P^\sharp(\alpha) := P(\alpha, \beta)$. In the symplectic case $P^\sharp(N^*C) = TC^\perp$ is the orthogonal with respect to the symplectic form ω of the tangent space to C.
The characteristic distribution defined by $P^\sharp(N^*C)$ is involutive; it is spanned at each point by the Hamiltonian vector fields corresponding to functions which are locally in $\mathcal{J}_C$.

We assume the canonical foliation to have a nice leaf space M_{red} (i.e. a structure of a smooth manifold such that the canonical projection $\pi : C \longrightarrow M_{red}$ is a submersion). Then M_{red} is a Poisson manifold in a canonical way: one defines the normalizer of $\mathcal{J}_C$

$$\mathcal{B}_C = \{f \in C^\infty(M) \mid \{f, \mathcal{J}_C\} \subseteq \mathcal{J}_C\},$$

$$\text{then} \qquad \mathcal{B}_C/\mathcal{J}_C \ni [f] \mapsto \iota^* f \in \pi^* C^\infty(M_{red})$$

induces an isomorphism of Poisson algebras.

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Reduction in Deformation Quantization following Bordemann, Herbig,,Waldmann

The associative algebra $\underline{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ is playing the role of the quantized observables of the big system.

A good analog of the vanishing ideal $\mathcal{J}_C$ will be a left ideal $\underline{\mathcal{J}}_C \subseteq C^\infty(M)[[\lambda]]$ such that the quotient $C^\infty(M)[[\lambda]]/\underline{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to the functions $C^\infty(C)[[\lambda]]$ on C.

Then one defines

$$\underline{\mathcal{B}}_C = \{a \in \underline{\mathcal{A}} \mid [a, \underline{\mathcal{J}}_C] \subseteq \underline{\mathcal{J}}_C\},$$

and considers the associative algebra

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as the reduced algebra $\underline{\mathcal{A}}_{red}$. Of course, one need then to show that $\underline{\mathcal{B}}_C / \underline{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to $C^\infty(M_{red})[[\lambda]]$ in such a way, that the isomorphism induces a star product $\star_{red}$ on M_{red}.

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Marsden Weinstein reduction

We shall start from a strongly invariant star product on M, and consider here the particular case of the Marsden-Weinstein reduction: let $\mathsf{L} : G \times M \longrightarrow M$ be a smooth left action of a connected Lie group G on M by Poisson diffeomorphisms and assume we have an ad^*-equivariant momentum map J.
The constraint manifold C is chosen to be the level surface of J for momentum $0 \in \mathfrak{g}^*$ (we assume, for simplicity, that 0 is a regular value) :

$$C = J^{-1}(\{0\})$$

(it is an embedded submanifold which is coisotropic).
The group G acts on C and the reduced space is the orbit space of this group action of G on C.
In order to guarantee a good quotient we assume that G acts freely and properly and we assume that G acts properly not only on C but on all of M.
In this case we can find an open neighbourhood $M_{\mathrm{nice}} \subseteq M$ of C with the following properties: there exists a G-equivariant diffeomorphism

$$\Phi : M_{\mathrm{nice}} \longrightarrow U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^*$$

onto an open neighbourhood U_{nice} of $C \times \{0\}$, where the G-action on $C \times \mathfrak{g}^*$ is the product action of the one on C and Ad^*, such that for each $p \in C$ the subset $U_{\mathrm{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped around the origin $\{p\} \times \{0\}$ and the momentum map J is given by the projection onto the second factor, i.e. $J_{|M_{\mathrm{nice}}} = \mathrm{pr}_2 \circ \Phi$.

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classical Koszul resolution

BRST is a technique to describe the functions on the reduced space and was used in the theory of reduction in deformation quantization Bordemann, Herbig and Waldmann; a simpler description is the **Koszul complex** : $\left(C^\infty(M_{\mathrm{nice}}, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}) = C^\infty(M_{\mathrm{nice}}) \otimes \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}\,,\, \partial\right)$,

∂ being the *Koszul differential* $\partial x = \mathrm{i}(J)x = \sum_a J_a\, \mathrm{i}(e^a)x$ with $\{e_a\}$ a basis of $\mathfrak{g}$.

Defining the *prolongation map* : $C^\infty(C) \ni \phi \mapsto \mathrm{prol}(\phi) = (\mathrm{pr}_1 \circ \Phi)^* \phi \in C^\infty(M_{\mathrm{nice}})$, and the *homotopy* : $C^\infty(M_{\mathrm{nice}}, \Lambda^k_{\mathbb{C}}\mathfrak{g}) \ni x \mapsto (h_k x)(p) = e_a \wedge \int_0^1 t^k \frac{\partial(x\circ\Phi^{-1})}{\partial \mu_a}(c, t\mu)\, \mathrm{d}\, t$,
one shows that $\mathrm{prol}\,\iota^* + \partial_1 h_0 = \mathrm{Id}_{C^\infty(M_{\mathrm{nice}})}$, $\iota^*\partial_1 = 0$, thus $\mathcal{J}_C = \ker \iota^* = \mathrm{Im}\,\partial_1$ and

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Deformation of the Koszul complex

Let $\star$ be a strongly invariant bidifferential formal star product (Recall that a star product is strongly invariant if it is invariant and $J_\xi \star f - f \star J_\xi = \nu\{J_\xi, f\} = \nu\, \mathcal{L}_{\xi^{*M}} f$ for all $f \in C^\infty(M)[[\nu]]$ and $\xi \in \mathfrak{g}$.) on M, so that we start from the "big" algebra of quantized observables

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To define the left ideal, one first **deforms the Koszul complex**, introducing

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The deformed left ideal

All those maps are G-invariant, $\underline{\mathbf{h}}_0^{(\kappa)}\,\mathrm{prol} = 0$, $\underline{\iota}_\kappa^*\underline{\partial}_1^{(\kappa)} = 0$, $\underline{\iota}_\kappa^*\,\mathrm{prol} = \mathsf{Id}_{C^\infty(C)[[\nu]]}$, and $\mathrm{prol}\,\underline{\iota}_\kappa^* + \underline{\partial}_1^{(\kappa)}\underline{\mathbf{h}}_0^{(\kappa)} = \mathsf{Id}_{C^\infty(M_{\mathrm{nice}})[[\nu]]}$ as well as $\underline{\mathbf{h}}_{k-1}^{(\kappa)}\underline{\partial}_k^{(\kappa)} + \underline{\partial}_{k+1}^{(\kappa)}\underline{\mathbf{h}}_k^{(\kappa)} = \mathsf{Id}_{C^\infty(M,\Lambda^k_{\mathbb{C}}\mathfrak{g})[[\nu]]}$.

One defines the **deformed left star ideal**

$$\underline{\mathcal{J}}_C = \mathrm{im}\,\underline{\partial}_1^{(\kappa)} = \ker \underline{\iota}_\kappa^*.$$

The left module $C^\infty(M_{\mathrm{nice}})[[\nu]]/\underline{\mathcal{J}}_C$ is isomorphic to $C^\infty(C)[[\nu]]$ with module structure $\bullet_\kappa$ defined by $f \bullet_\kappa \phi = \underline{\iota}_\kappa^*(f \star \mathrm{prol}(\phi))$ for $\phi \in C^\infty(C)[[\nu]], f \in C^\infty(M_{\mathrm{nice}})[[\nu]$ via the map

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whose inverse is $\phi \mapsto [\mathrm{prol}(\phi)]$. This left module structure is G-invariant $(\mathsf{L}_g^*(f \bullet_\kappa \phi) = (\mathsf{L}_g^* f) \bullet_\kappa (\mathsf{L}_g^*\phi)$ for all $g \in G$, $f \in C^\infty(M)[[\nu]]$, and $\phi \in C^\infty(C)[[\nu]])$ and for all $\xi \in \mathfrak{g}$ one has, using the fact that the star product is strongly invariant,
$J_\xi \bullet_\kappa \phi = \nu\,\mathcal{L}_{\xi * C}\,\phi - \nu\kappa\Delta(\xi)\phi$.

The deformed left ideal

All those maps are G-invariant, $\underline{\mathbf{h}}_0^{(\kappa)} \operatorname{prol} = 0$, $\underline{\iota}_\kappa^* \underline{\partial}_1^{(\kappa)} = 0$, $\underline{\iota}_\kappa^* \operatorname{prol} = \mathsf{Id}_{C^\infty(C)[[\nu]]}$, and $\operatorname{prol} \underline{\iota}_\kappa^* + \underline{\partial}_1^{(\kappa)} \underline{\mathbf{h}}_0^{(\kappa)} = \mathsf{Id}_{C^\infty(M_{\mathrm{nice}})[[\nu]]}$ as well as $\underline{\mathbf{h}}_{k-1}^{(\kappa)} \underline{\partial}_k^{(\kappa)} + \underline{\partial}_{k+1}^{(\kappa)} \underline{\mathbf{h}}_k^{(\kappa)} = \mathsf{Id}_{C^\infty(M, \Lambda^k_{\mathbb{C}} \mathfrak{g})[[\nu]]}$.

One defines the **deformed left star ideal**

$$\underline{\mathcal{J}}_C = \operatorname{im} \underline{\partial}_1^{(\kappa)} = \ker \underline{\iota}_\kappa^* .$$

The left module $C^\infty(M_{\mathrm{nice}})[[\nu]]/\underline{\mathcal{J}}_C$ is isomorphic to $C^\infty(C)[[\nu]]$ with module structure $\bullet_\kappa$ defined by $f \bullet_\kappa \phi = \underline{\iota}_\kappa^*(f \star \operatorname{prol}(\phi))$ for $\phi \in C^\infty(C)[[\nu]], f \in C^\infty(M_{\mathrm{nice}})[[\nu]$ via the map

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The reduced star product

One considers its **normalizer** $\underline{\mathcal{B}}_C = \{f \in C^\infty(M)[[\nu]] \mid [f, \underline{\mathcal{J}}_C]_\star \subseteq \underline{\mathcal{J}}_C\}$.

For a g in $\underline{\mathcal{J}}_C$, $g = g^a \star J_a + \nu\kappa C^a_{ba} g^b$ with $g^a \in C^\infty(M)[[\nu]]$; for $f \in C^\infty(M)[[\nu]]$, the $\star$-bracket is $[f,g]_\star = \underline{\partial}_1^{(\kappa)} h - \nu g^a \star \mathcal{L}_{(e_a)_{*M}} f$ with $h = (f \star g_a)e^a \in C^\infty(M, \mathfrak{g})$.

Thus $[f,g]_\star$ is in $\underline{\mathcal{J}}_C$ iff $g^a \star \mathcal{L}_{(e_a)_{*M}} f$ is in the image of $\underline{\partial}_1^{(\kappa)}$ for all g^a.

This shows that $f \in \underline{\mathcal{B}}_C$ iff $\mathcal{L}_{\xi_{*M}} f \in \operatorname{im} \underline{\partial}_1^{(\kappa)} = \ker \underline{\iota}_\kappa^*$ thus iff $\mathcal{L}_{\xi_{*C}}\, \underline{\iota}_\kappa^* f = 0$ for all $\xi \in \mathfrak{g}$, i.e. iff $\underline{\iota}_\kappa^* f \in \pi^* C^\infty(M_{red})[[\nu]]$.

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whose inverse is $u \mapsto [\mathrm{prol}(\pi^* u)]$.

The **reduced star** product $\star_{red}^{(\kappa)}$ on $C^\infty(M_{red})[[\nu]]$ is induced from $\underline{\mathcal{B}}_C / \underline{\mathcal{J}}_C$ and explicitly given by

$$\pi^*(u \star_{red}^{(\kappa)} v) = \underline{\iota}_\kappa^* \left(\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v)\right).$$

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One checks that it is given by a series of bidifferential operators.

Involutions

In quantum mechanics, the algebra of quantum observables has a *-involution given in the usual picture, where observables are represented by operators, by the passage to the adjoint operator.

In deformation quantization, a *-involution on $\underline{\mathcal{A}} = (C^\infty(M)[[\nu]], \star)$ for $\nu = i\lambda$ (with $\lambda \in \mathbb{R}$) may be obtained, asking the star product to be Hermitian, i.e such that $\overline{f \star g} = \overline{g} \star \overline{f}$ and the *-involution is then complex conjugation.

With Stefan Waldmann, we have studied how to get in a natural way a *-involution for the reduced algebra, assuming that $\star$ is a Hermitian star product on M.

The main idea here is to use a representation of the reduced quantum algebra and to translate the notion of the adjoint.

Observe that $\underline{\mathcal{B}}/\underline{\mathcal{J}}$ can be identified (with the opposite algebra structure) to the algebra of $\underline{\mathcal{A}}$-linear endomorphisms of $\underline{\mathcal{A}}/\underline{\mathcal{J}}$.

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Observe that $\underline{\mathcal{B}}/\underline{\mathcal{J}}$ can be identified (with the opposite algebra structure) to the algebra of $\underline{\mathcal{A}}$-linear endomorphisms of $\underline{\mathcal{A}}/\underline{\mathcal{J}}$.

Reduced involutions

We use an additional positive linear functional $\omega : \underline{\mathcal{A}} \longrightarrow \mathbb{C}[[\lambda]]$ such that the Gel'fand ideal of ω, $\underline{\mathcal{J}}_\omega = \{a \in \underline{\mathcal{A}} \mid \omega(a^* a) = 0\}$, coincides with the left ideal $\underline{\mathcal{J}}$ used in reduction, and such that all left $\underline{\mathcal{A}}$-linear endomorphisms of the space of the GNS representation $\mathcal{H}_\omega = \underline{\mathcal{A}}/\underline{\mathcal{J}}_\omega$, with the pre Hilbert space structure defined via $\langle \psi_a, \psi_b \rangle = \omega(a^* b)$, are adjointable.

Then the algebra of $\underline{\mathcal{A}}$-linear endomorphisms of $\mathcal{H}_\omega$ (with the opposite structure) is equal to $\underline{\mathcal{B}}/\underline{\mathcal{J}}_\omega$ so that $\underline{\mathcal{B}}/\underline{\mathcal{J}}$ becomes in a natural way a *-subalgebra of the set $\mathfrak{B}(\mathcal{H}_\omega)$ of adjointable maps.

A formal series of smooth densities $\sum_{r=0}^{\infty} \lambda^r \mu_r \in \Gamma^\infty(|\Lambda^{\mathrm{top}}| T^* C)[[\nu]]$ on the coisotropic submanifold C, such that $\overline{\mu} = \mu$ is real, $\mu_0 > 0$ and so that μ transforms under the G-action as $\mathsf{L}^*_{g^{-1}} \mu = \frac{1}{\Delta(g)} \mu$ (where Δ is the modular function), yields a positive linear functional which defines a *-involution on the reduced space.

In the classical Marsden Weinstein reduction, complex conjugation is a *-involution of the reduced quantum algebra. Looking whether the *-involution corresponding to a series of densities μ is the complex conjugation yields a new notion of quantized modular class.

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Representations of the reduced star product

We also studied representations of the reduced algebra with the *-involution given by complex conjugation, relating the categories of modules of the big algebra and the reduced algebra.

The usual technique to relate categories of modules is to use a bimodule and the tensor product to pass from modules of one algebra to modules of the other.

The construction of the reduced star products gives a bimodule structure on $C^\infty(C)[[\nu]]$. The space of formal series $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\nu]]$ where

$$\mathcal{C}^\infty_{\mathrm{cf}}(C) = \left\{\phi \in C^\infty(C) \;\middle|\; \mathrm{supp}(\phi) \cap \pi^{-1}(K) \text{ is compact for all compact} K \subseteq M_{red}\right\}$$

is a left $(C^\infty(M)[[\nu]], \star)$- and a right $(C^\infty(M_{red})[[\nu]], \star_{red})$-module; on this bimodule there is a $C^\infty(M_{red})[[\nu]]$-valued inner product.

This bimodule structure and inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\nu]]$ give a strong Morita equivalence bimodule between $C^\infty(M_{red})[[\nu]]$ and the finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\nu]]$.

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