$Spin^c$, Mp^c and symplectic Dirac operators

Michel Cahen and Simone Gutt

We are happy to dedicate our talks and this summary to the memory of Boris Fedosov; we chose a subject which is close to operators and quantization, two fields in which Boris Fedosov brought essential new contributions.

Abstract. We advertize the use of the group Mp^c (a circle extension of the symplectic group) instead of the metaplectic group (a double cover of the symplectic group). The essential reason is that Mp^c-structures exist on any symplectic manifold. They first appeared in the framework of geometric quantization[1, 2]. In a joint work with John Rawnsley [3], we used them to extend the definition of symplectic spinors and symplectic Dirac operators which were first introduced by Kostant [4] and K. Haebermann [5] in the presence of a metaplectic structure . We recall here this construction, stressing the analogies with the group $Spin^c$ in Riemannian geometry. Dirac operators are defined as a contraction of Clifford multiplication and covariant derivatives acting on spinor fields; in Riemannian geometry, the contraction is defined using the Riemannian structure. In symplectic geometry one contracts using the symplectic structure or using a Riemannian structure defined by the choice of a positive compatible almost complex structure. We suggest here more general contractions yielding new Dirac operators.

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1. The group $Spin^c$

On an oriented Riemannian manifold of dimension m, (M, g), the tangent space at any point $x \in M$, T_xM, is modelled on a Euclidean vector space $(V, \tilde{g})$.

The isomorphism group of this model is the special orthogonal group

$$SO(V, \tilde{g}) := \{A \in Gl(V) \,|\, \tilde{g}(Au, Av) = \tilde{g}(u, v) \ \det A = 1\,\}$$

and there exists a natural principal bundle $\mathcal{B}(M, g) \xrightarrow{p} M$, with structure group $SO(V, \tilde{g})$, which is the bundle of oriented orthonormal frames of the

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tangent bundle; the fiber above $x \in M$, $p^{-1}(x)$, consists of all linear isomorphisms of Euclidean spaces preserving the orientation

$$f : (V, \tilde{g}) \to (T_x M, g_x)$$

with $SO(V, \tilde{g})$ acting on the right on $\mathcal{B}(M, g)$ by composition

$$f \cdot A := f_\circ A \quad \forall f \in \mathcal{B}(M, g), A \in SO(V, \tilde{g}).$$

The tangent bundle TM is associated to the bundle of oriented orthonormal frames for the standard representation st of the structure group $SO(V, \tilde{g})$ on V (i.e. $st(A)v = Av$),

$$TM = \mathcal{B}(M, g) \times_{(SO(V,\tilde{g}),st)} V$$

with

$$f(v) \simeq [(f, v)] = [(f_\circ A, A^{-1}v)].$$

The **Clifford Algebra** $Cl(V, \tilde{g})$ is the associative unital algebra generated by V such that $u \cdot v + v \cdot u = -2\tilde{g}(u, v)1$ for all $u, v \in V$. To simplify notations, we shall assume here to be in the even dimensional situation $m = 2n$. Then the complexification of the Clifford algebra is identified with the space of complex linear endomorphisms of the exterior algebra built on a maximal isotropic subspace $W \subset V^{\mathbb{C}} = W \oplus \overline{W}$

$$Cl(V, \tilde{g})^{\mathbb{C}} \simeq \mathrm{End}(\Lambda W).$$

Indeed, one associates to an element $w \in W$ the endomorphism

$$cl(w)\alpha = \sqrt{2}\, w \wedge \alpha \qquad \forall \alpha \in \Lambda W$$

and to the conjugate element $\overline{w} \in \overline{W}$ the endomorphism

$$cl(\overline{w})\alpha = -\sqrt{2}\, i(\underline{\overline{w}}_{\tilde{g}})\alpha \qquad \forall \alpha \in \Lambda W, \quad \text{with} \quad \underline{v}_{\tilde{g}}(v') = \tilde{g}(v, v').$$

The **Spinor space** S is a complex vector space with a Hermitian scalar product $< \cdot, \cdot >$, carrying an irreducible representation cl of $Cl(V, \tilde{g})$, so that each element of V acts in a skewhermitian way

$$< cl(v)\alpha, \beta > + < \alpha, cl(v)\beta >= 0.$$

Here $S = \Lambda W$, and the Hermitian scalar product is the natural extension of

$$< w, w' >= \tilde{g}(w, \overline{w'}) \qquad w, w' \in W.$$

On a Riemannian manifold (M, g), one defines -when possible- a spinor bundle $\mathcal{S}(M, g)$ and a fiberwise Clifford multiplication Cl of the tangent bundle TM acting on the spinor bundle $\mathcal{S}(M, g)$ by gluing the above construction. For this, one needs :

- a principal bundle $\mathcal{B} \xrightarrow{p_\mathcal{B}} M$ with structure group G ;

- a group homomorphism $\sigma : G \to SO(V, \tilde{g})$ so that the tangent space is associated to $\mathcal{B}$ for the representation $st_\circ\sigma$ of G on V:

$$TM = \mathcal{B} \times_{(G,\sigma)} V;$$

it is equivalent to ask for the existence of a map

$$\Phi : \mathcal{B} \to \mathcal{B}(M, g)$$

which is fiber-preserving (i.e. $p_{\mathcal{B}} = p_\circ \Phi$) and (G,σ)-equivariant , i.e.

$$\Phi(\tilde{f}\cdot\tilde{A}) = \Phi(\tilde{f})\cdot\sigma(\tilde{A}) \qquad \forall \tilde{f}\in\mathcal{B},\ \tilde{A}\in G;$$

- a unitary representation r of G on the spinor space S to define **the spinor bundle**

$$\mathcal{S}(M,g) = \mathcal{B}\times_{(G,r)} S.$$

- The Clifford multiplication is well defined via

$$Cl([(\tilde{f},v)])\left([(\tilde{f},s)]\right) := [(\tilde{f}, cl(v)s)] \quad \forall \tilde{f}\in\mathcal{B}, v\in V, s\in S$$

if and only if $cl(\sigma(\tilde{A})v)\left(r(\tilde{A})s\right) = r(\tilde{A})\,(cl(v)s)$, i.e. iff

$$cl(\sigma(\tilde{A})v) = r(\tilde{A})_\circ cl(v)_\circ r(\tilde{A})^{-1} \quad \forall \tilde{A}\in G, v\in V. \tag{1}$$

There is no representation r of $SO(V,\tilde{g})$ on S satisfying (1). Indeed, any representation r of G on S is given by a homomorphism n of G into $Cl(V,\tilde{g})^{\mathbb{C}} \simeq \mathrm{End}(\Lambda W) = \mathrm{End}(S)$ and condition (1) is equivalent to

$$\sigma(\tilde{A})v = n(\tilde{A})\cdot v\cdot n(\tilde{A})^{-1}.$$

The differential of n yields a homomorphism n_* of the Lie algebra $\mathfrak{g}$ of G into $Cl(V,\tilde{g})^{\mathbb{C}}$ endowed with the bracket

$$[\alpha,\beta]_{Cl} = \alpha\cdot\beta - \beta\cdot\alpha$$

and condition (1) yields

$$[n_*(B), v]_{Cl} = \sigma_* Bv.$$

The Lie algebra $\mathfrak{so}(V,\tilde{g})$ naturally embeds in $Cl(V,\tilde{g})$:

$$\nu(\underline{v}_{\tilde{g}}\otimes w - \underline{w}_{\tilde{g}}\otimes v) = \frac{1}{4}(v\cdot w - w\cdot v) \quad \text{with} \quad \underline{w}_{\tilde{g}}(v) = \tilde{g}(w,v)$$

and this satisfies :

$$[\nu(B), v]_{Cl} = Bv \quad [\nu(B),\nu(B')]_{Cl} = \nu([B,B'])$$

but this does not lift to a homomorphism of $SO(V,\tilde{g})$ into $Cl(V,\tilde{g})^{\mathbb{C}}$ since $\exp\left(2\pi\nu(\underline{e_1}_{\tilde{g}}\otimes e_2 - \underline{e_2}_{\tilde{g}}\otimes e_1)\right) = -\mathrm{Id}$ for e_1, e_2 two orthonormal vectors.

One way to proceed is to define **the group** $Spin$ as the connected subgroup of $Gl(S)$ with Lie algebra $\nu(\mathfrak{so}(V,\tilde{g}))$; it is a double cover of $SO(V,\tilde{g})$. A pair consisting of a principal $Spin$ bundle $\mathcal{B}\xrightarrow{p_{\mathcal{B}}} M$ and a map $\Phi:\mathcal{B}\to\mathcal{B}(M,g)$ which is fiber-preserving and $(Spin,\sigma)$-equivariant is called a **Spin structure** on the manifold M. Such a structure only exists if a cohomology class (the second Stiefel Whitney class) vanishes.

Another way to proceed is to stress the importance of the fundamental equation (1) and to consider all unitary transformations of S for which this equation is satisfied. More precisely, for any $A\in SO(V,\tilde{g})$, the maps $cl(v)$ and $cl(Av)$ extend to two representations of $Cl(V,\tilde{g})$ on the spinor space S which are equivalent and one defines **the group** $Spin^c$ as the set of unitary intertwiners between those representations:

$$Spin^c = \{(U,A)\in\mathcal{U}(S)\times SO(V,\tilde{g})\,|\,Ucl(v)U^{-1} = cl(Av)\,\forall v\in V\}$$

with multiplication defined componentwise, the natural unitary representation r on S defined by $r(U,A)=U$, and the obvious homomorphism

$$\sigma : Spin^{\mathrm{c}} \to SO(V,\tilde g) : (U,A) \to A$$

with kernel $U(1)$. This defines a short exact sequence:

$$1 \to U(1) \xrightarrow{i} Spin^{\mathrm{c}} \xrightarrow{\sigma} SO(V,\tilde g) \to 1 \tag{2}$$

which does not split.
At the level of Lie algebras, the sequence splits and

$$\mathfrak{spin}^{\mathrm{c}} = \nu(\mathfrak{so}(V,\tilde g)) \oplus \mathfrak{u}(1).$$

Observe that $[cl(\nu(\sigma_* X)), cl(v)] = cl(\sigma_*(X)v) = [r_*X, cl(v)]$ for all X in $\mathfrak{spin}^{\mathrm{c}}$ so that $r_*X - cl(\nu(\sigma_*X)) = c(X)\mathrm{Id}$ where c is a unitary infinitesimal character of $\mathfrak{spin}^{\mathrm{c}}$.
This presentation of the group $Spin^{\mathrm{c}}$ as the set of intertwiners can be generalised for any signature in any dimension. To relate this to the $Spin$ group (which historically came first), let us observe that

$$Spin^{\mathrm{c}} = (Spin \times U(1)) /_{\pm\{\mathrm{Id}\}}$$

(and this is the usual presentation of the group $Spin^{\mathrm{c}}$). The character

$$\eta : Spin^{\mathrm{c}} \to U(1) : [(A,\lambda)] \mapsto \lambda^2 \qquad \forall A \in Spin, \lambda \in U(1)$$

is the squaring map on the central $U(1)$ and has for kernel $Spin$. We have

$$r_*X = cl(\nu(\sigma_*X)) + \tfrac{1}{2}\eta_*(X)\mathrm{Id} \qquad \forall X \in \mathfrak{spin}^{\mathrm{c}}.$$

Observe that the short exact seqence 2 splits over the unitary group; indeed, given a complex structure j on V which is an isometry for the metric $\tilde g$, the unitary group $U(V,\tilde g, j)$ – which is identified to the subgroup of j-linear endomorphisms in $SO(V,\tilde g)$ – injects into $Spin^{\mathrm{c}}$ in the following way. Let us choose $W \subset V^{\mathbb{C}}$ to be the $+i$ eigenspace for j in $V^{\mathbb{C}}$ and realize $Spin^{\mathrm{c}}$ in the group of unitary endomorphisms of $S = \Lambda W$ as above; an element $U \in U(V,\tilde g, j)$ acting on $V^{\mathbb{C}}$ stabilizes W and induces a unitary endomorphism denoted $\Lambda(U)$ of ΛW; one has

$$\left.\begin{array}{l} \Lambda(U)cl(w)\Lambda(U)^{-1} = cl(Uw) \\ \Lambda(U)cl(\overline{w})\Lambda(U)^{-1} = cl(U\overline{w}) \end{array}\right\} \forall w \in W$$

so that $\Lambda(U)$ is in $Spin^{\mathrm{c}}$ and we have the injection:

$$\Lambda : U(V,\tilde g, j) \to Spin^{\mathrm{c}} : U \mapsto \Lambda(U).$$

At the level of Lie algebras, we have:

$$\Lambda_*(X) = cl(\nu(X)) + \tfrac{1}{2}\mathrm{Trace}_j(X)\mathrm{Id} \qquad \forall X \in \mathfrak{u}(V,\tilde g, j).$$

Let $SpU^{\mathrm{c}}(V,\tilde g, j)$ be the inverse image of $U(V,\tilde g, j)$ under σ and define the character

$$\lambda : SpU^{\mathrm{c}}(V,\tilde g, j) \to U(1) : \Lambda(U)\tilde\lambda \mapsto \tilde\lambda \qquad \forall U \in U(V,\tilde g, j), \tilde\lambda \in U(1).$$

The isomorphism

$$\sigma \times \lambda : SpU^{\mathrm{c}}(V,\tilde g, j) \to U(V,\tilde g, j) \times U(1)$$

has inverse $\Lambda \times i$. The complex determinant defines another character $\det_j \circ \sigma$ on $SpU^c(V, \tilde{g}, j)$ and the three characters are related by

$$\eta = \lambda^2 \det_j \circ \sigma.$$

A pair consisting of a principal $Spin^c$ bundle $\mathcal{B} \xrightarrow{p_{\mathcal{B}}} M$ and a fiber-preserving and $(Spin^c, \sigma)$-equivariant map $\Phi : \mathcal{B} \to \mathcal{B}(M, g)$ is called a $Spin^c$ **structure** on the manifold M. Such a structure does not always exist.
If it exists, it is not unique. Indeed given a $Spin^c$ structure $(\mathcal{B}, \Phi)$ and a $U(1)$ principal bundle $L^1 \xrightarrow{p_1} M$, one builds a new $Spin^c$ structure $(\mathcal{B}, \Phi)$ by

$$\mathcal{B}' = \left(\mathcal{B} \times_M L^1\right) \times_{(Spin^c \times U(1))} Spin^c$$

for the homomorphism of $Spin^c \times U(1)$ into $Spin^c$ given by the natural injection i of $U(1)$ and multiplication, and by

$$\Phi' : \mathcal{B}' \to \mathcal{B}(M, g) : [((\tilde{f}, s), (U, A))] \mapsto \Phi(\tilde{f}) \cdot A.$$

Since 2 splits over $U(V, \tilde{g}, j)$, a Riemannian manifold (M, g) which admits an almost complex structure J so that $g(JX, JY) = g(X, Y)$, always admits $Spin^c$-structures. In particular Kähler manifolds always admit $Spin^c$-structures. Denoting by $\mathcal{B}(M, g, J) \subset \mathcal{B}(M, g)$ the $U(V, \tilde{g}, j)$ principal bundle of complex orthonormal frames of TM then

$$\mathcal{B} := \mathcal{B}(M, g, J) \times_{U(V, \tilde{g}, j)} Spin^c$$

and

$$\Phi : \mathcal{B}(M, g, J) \times_{U(V, \tilde{g}, j)} Spin^c \to \mathcal{B}(M, g) : [(f, (U, A)] \mapsto f \cdot A$$

define a $Spin^c$ structure on (M, g).

2. The Symplectic Clifford Algebra and the group Mp^c

On a symplectic manifold (M, ω) of dimension $m = 2n$, the tangent space at any point is modelled on a symplectic vector space (V, Ω).
The isomorphism group of this model is the symplectic group

$$Sp(V, \Omega) := \{A \in Gl(V) \mid \Omega(Au, Av) = \Omega(u, v) \, \forall u, v \in V\}$$

and there exists a natural principal bundle $\mathcal{B}(M, \omega) \xrightarrow{p} M$, with structure group $Sp(V, \Omega)$, which is the bundle of symplectic frames of the tangent bundle; the fiber above $x \in M$, consists of all linear isomorphisms of symplectic spaces

$$f : (V, \Omega) \to (T_x M, \omega_x)$$

with $Sp(V, \Omega)$ acting on the right on $\mathcal{B}(M, \omega)$ by composition. The tangent bundle TM is associated to the bundle of symplectic frames for the standard representation st of the structure group $Sp(V, \Omega)$ on V:

$$TM = \mathcal{B}(M, \omega) \times_{(Sp(V, \Omega), st)} V.$$

The **symplectic Clifford Algebra** $Cl(V,\Omega)$ is the associative unital algebra generated by V such that

$$u\cdot v - v\cdot u = \frac{i}{\hbar}\Omega(u,v)1 \qquad \forall u,v\in V.$$

The Heisenberg group is $H(V,\Omega)=V\times\mathbb{R}$ with multiplication defined by $(v_1,t_1)(v_2,t_2)=(v_1+v_2,t_1+t_2-\frac{1}{2}\Omega(v_1,v_2))$; its Lie algebra is the space $\mathfrak{h}(V,\Omega)=V\oplus\mathbb{R}$ with brackets $[(v,\alpha),(w,\beta)]=(0,\,-\Omega(v,w))$, so that a representation of $Cl(V,\Omega)$ is a representation of $\mathfrak{h}(V,\Omega)$ with central character equal to $-\frac{i}{\hbar}$.

Let $\mathcal{H},\rho$ be an irrededucible unitary representation of the Heisenberg group $H(V,\Omega)$ on a complex separable Hilbert space $\mathcal{H}$ with central character $\rho(0,t)=e^{-\frac{i}{\hbar}t}\mathrm{Id}$; by Stone-Von Neumann's theorem, such a representation is unique up to unitary equivalence. The space $\mathcal{H}^\infty$ of smooth vectors of this representation (or its dual $\mathcal{H}^{-\infty}$) is the **symplectic spinor space** S. It carries a Hermitian scalar product and the symplectic Clifford multiplication *scl* of V on S is given by the skewhermitian operators

$$scl(v):=\rho_*\,(v,0)\,.$$

To glue the above construction on a symplectic manifold and build a spinor bundle endowed with a symplectic Clifford multiplication by ele–ments of the tangent bundle, one uses, as before:

- a principal bundle $\mathcal{B}\xrightarrow{p_{\mathcal{B}}}M$ with structure group G;
- a group homomorphism $\sigma: G\to Sp(V,\Omega)$;
- a fiber-preserving (G,σ)–equivariant map $\Phi:\mathcal{B}\to\mathcal{B}(M,\omega)$, i.e. $p_{\mathcal{B}}=p_\circ\Phi$ and $\Phi(\tilde f\cdot\tilde A)=\Phi(\tilde f)\cdot\sigma(\tilde A)$ for all $\tilde f\in\mathcal{B}$, $\tilde A\in G$;
- a unitary representation r of G on $\mathcal{H}$ stabilizing the spinor space S and satisfying the fundamental equation

$$\rho(\sigma(\tilde A)v,0)=r(\tilde A)_\circ\rho(v,0)_\circ r(\tilde A)^{-1}\quad\forall\tilde A\in G, v\in V \tag{3}$$

which implies on S

$$scl(\sigma(\tilde A)v)=r(\tilde A)_\circ scl(v)_\circ r(\tilde A)^{-1}\quad\forall\tilde A\in G, v\in V.$$

The **symplectic spinor bundle** is then $\mathcal{S}=\mathcal{B}\times_{(G,r)}S$; and the **symplectic Clifford multiplication** $sCl: TM\times_M\mathcal{S}\to\mathcal{S}$ is given by

$$sCl([(\tilde f,v)])\Big([(\tilde f,s)]\Big):=[(\tilde f,scl(v)s)]\quad\forall\tilde f\in\mathcal{B}, v\in V, s\in S.$$

There is an embedding of the symplectic Lie algebra into the symplectic Clifford algebra given by

$$\nu(\underline{v}_\Omega\otimes w+\underline{w}_\Omega\otimes v)=\frac{-i\hbar}{2}(v\cdot w+w\cdot v)\quad\text{with}\quad\underline{w}_\Omega(v)=\Omega(w,v)$$

and one has

$$[\nu(B),v]_{sCl}=Bv\quad[\nu(B),\nu(B')]_{sCl}=\nu([B,B'])$$

but there is no lift to a representation of the symplectic group on $\mathcal{H}$ satisfying equation (3).

On the other hand, the symplectic group $Sp(V,\Omega)$ acts as automorphisms of the Heisenberg group via

$$A\cdot(v,t) := (Av,t).$$

Given an element $A\in Sp(V,\Omega)$, the two representations $\rho(v,t)$ and $\rho(Av,t)$ of the Heisenberg group on $\mathcal{H}$ have the same central character, so they are equivalent. The fundamental equation (3) states that $r(\tilde{A})$ is an intertwiner of these two representations. **The group** Mp^c is defined as the set of all such unitary intertwiners :

$$Mp^c = \{(U,A)\in\mathcal{U}(\mathcal{H})\times Sp(V,\Omega)\,|U\rho(v,0)U^{-1}=\rho(Av,0)\,\forall v\}$$

with multiplication defined componentwise, the natural unitary representation r on $\mathcal{H}$ defined by $r(U,A)=U$, and the obvious homomorphism

$$\sigma : Mp^c\to Sp(V,\Omega) : (U,A)\mapsto A$$

which has kernel $U(1)$ by irredicibility of ρ. The short exact sequence

$$1\to U(1)\xrightarrow{i} Mp^c\xrightarrow{\sigma} Sp(V,\Omega)\to 1 \tag{4}$$

does not split. At the level of Lie algebras, the sequence splits and

$$\mathfrak{mp}^c=\nu(\mathfrak{sp}(V,\Omega))\oplus\mathfrak{u}(1).$$

Observe that $[scl(\nu(\sigma_*X)),scl(v)]=scl(\sigma_*(X)v)=[r_*X,scl(v)]$ for all X in $\mathfrak{mp}^c$ so that $r_*X-scl(\nu(\sigma_*X))=c(X)\mathrm{Id}$ where c is a unitary infinitesimal character of $\mathfrak{mp}^c$.

2.1. Explicit description of the group Mp^c

To get the nice Fock's description of the irreducible unitary representation of the Heisenberg group with prescribed central character, one chooses a positive compatible complex structure on (V,Ω).

A **compatible complex structure** j is a (real) linear map of V which is symplectic, $\Omega(jv,jw)=\Omega(v,w)$, and satisfies $j^2=-I_V$. Given such a j, the map $(v,w)\mapsto\Omega(v,jw)$ is a non-degenerate symmetric bilinear form and j is **positive** if this form is positive definite.

The choice of j gives V the structure of a complex vector space, with $(x+iy)v=xv+yj(v)$ and when j is positive, one has a Hermitean structure on V defined by

$$\langle v,w\rangle_j=\Omega(v,jw)-i\Omega(v,w),\qquad |v|_j^2=\langle v,v\rangle_j.$$

The set of positive compatible j's is the contractible homogeneous space $Sp(V,\Omega)/U(V,\Omega,j)$.

Having chosen a positive compatible j, the Fock's description of the Hilbert space carrying an irreducible unitary representation ρ_j of the Heisenberg group with central character $\rho_j(0,t)=e^{-\frac{i}{\hbar}t}\mathrm{Id}$ is denoted $\mathcal{H}(V,\Omega,j)$; it is the space of holomorphic functions $f(z)$ on (V,Ω,j) which are L^2 in the sense of the norm $\|f\|_j$ given by

$$\|f\|_j^2=h^{-n}\int_V|f(z)|^2e^{-\frac{|z|_j^2}{2\hbar}}\,dz.$$

The unitary and irreducible action of the Heisenberg group $H(V,\Omega)$ on $\mathcal{H}(V,\Omega,j)$ is given by

$$(\rho_j(v,t)f)(z) = e^{-it/\hbar + \langle z,v\rangle_j/2\hbar - |v|_j^2/4\hbar} f(z-v).$$

The space $\mathcal{H}(V,\Omega,j)$ has the nice property to possess a family of **coherent states** e_v parametrised by V :

$$(e_v)(z) = e^{\frac{1}{2\hbar}\langle z,v\rangle_j} \quad \text{such that} \quad f(z) = (f,e_z)_j$$

so that any unitary operator U on $\mathcal{H}(V,\Omega,j)$ is entirely characterized by its Berezin kernel

$$(Ue_v,e_w)_j$$

which is a holomorphic function in w and an antiholomorphic function in v. The Heisenberg Lie algebra $\mathfrak{h}(V,\Omega)$ acts on smooth vectors and these include the coherent states; the Clifford multiplication is defined as $scl(v) = \dot\rho_j(v,0)$ and it splits:

$$(scl(v)f)(z) = \frac{1}{2\hbar}\langle z,v\rangle_j f(z) - (\partial_z f)(v) =: (c(v)f)(z) - (a(v)f)(z)$$

in a creation and an annihilation operators $c(v)$ and $a(v)$ which respectively raises and lowers the degree of a polynomial in z.

To give a description of the kernel of an element in the group Mp^{c}, we introduce a parametrization of the symplectic group. Writing an element $g \in Sp(V,\Omega)$ as $g = C_g + D_g$ with $C_g = \frac{1}{2}(g - jgj)$ the j-linear part and $D_g = \frac{1}{2}(g + jgj)$ the j-antilinear part, one observes that C_g is invertible and set $Z_g = C_g^{-1}D_g$.

The Berezin kernel $U(z,v) := (Ue_v,e_z)_j$ of the unitary operator U when $(U,g) \in Mp^{\mathrm{c}}(V,\Omega,j)$ is given by:

$$U(z,v) = \lambda \exp\frac{1}{4\hbar}\{2\langle C_g^{-1}z,v\rangle_j - \langle z, Z_{g^{-1}}z\rangle_j - \langle Z_g v,v\rangle_j\}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda^2 \det C_g| = 1$.

Indeed, since $(e_v)(w) = e^{\frac{1}{2\hbar}\langle w,v\rangle_j} = e^{\frac{1}{4\hbar}\langle v,v\rangle_j}(\rho_j(v,0)e_0)(w)$, we have $U(z,v)$ $= e^{\frac{1}{4\hbar}(\langle z,z\rangle_j + \langle v,v\rangle_j)}\,(U\rho_j(v,0)e_0, \rho_j(z,0)e_0)_j$.

For $v = g^{-1}z$, $U\rho_j(g^{-1}z,0) = \rho_j(z,0)U$ by definition of $Mp^{\mathrm{c}}(V,\Omega,j)$, and we get $U(z,g^{-1}z) = e^{\frac{1}{4\hbar}(\langle z,z\rangle_j + \langle g^{-1}z,g^{-1}z\rangle_j)}\,(Ue_0,e_0)_j$.

But $U(z,v)$ is holomorphic in z and antiholomorphic in v, so it is completely determined by its values for $(z, v = g^{-1}z)$. The formula given above follows from the fact that $v = g^{-1}z = C_{g^{-1}}(\mathrm{Id} + Z_{g^{-1}})z$ yields $z = (C_{g^{-1}})^{-1}v - Z_{g^{-1}}z$. We say that g,λ are the parameters of (U,g) when the kernel of U is given as above.

The product in $Mp^{\mathrm{c}}(V,\Omega,j)$ of (U_i,g_i) with parameters g_i,λ_i, $i = 1,2$ has parameters g_1g_2, λ_{12} with

$$\lambda_{12} = \lambda_1\lambda_2 e^{-\frac{1}{2}a\left(1 - Z_{g_1}Z_{g_2^{-1}}\right)}$$

where $a : GL(V,j)_+ := \{g \in GL(V,j) \mid g + g^* \text{ is positive definite}\} \to \mathbb{C}$ is the unique smooth function defined on the simply connected space $GL(V,j)_+$ such that

$$\det_j g = e^{a(g)} \qquad \text{and} \quad a(I) = 0$$

where $\det_j$ is the complex determinant of a j-linear endomorphism viewed as a complex endomorphism.
This proves that the group $Mp^c(V,\Omega,j)$ is a Lie group.

2.2. Character and subgroups of $Mp^c(V,\Omega,j)$

The group $Mp^c(V,\Omega,j)$ admits a **character** η given by

$$\eta(U,g) = \lambda^2 \det_j C_g$$

which is the squaring map on the central $U(1)$. The **metaplectic group** is the kernel of η; it is given by

$$Mp(V,\Omega,j) = \{(U,g) \in Mp^c(V,\Omega,j) \mid \lambda^2 \det_j C_g = 1\}.$$

It is a double covering of $Sp(V,\Omega)$ and

$$Mp^c(V,\Omega,j) = (Mp(V,\Omega,j) \times U(1)) /_{\pm\{\mathrm{Id}\}}.$$

We have:

$$r_* X = scl(\nu(\sigma_* X)) + \tfrac{1}{2}\eta_*(X)\mathrm{Id} \qquad \forall X \in \mathfrak{mp}^c.$$

An important fact is that the short exact seqence 4 splits over the unitary group. Indeed the unitary group $U(V,\Omega,j)$ injects into $Mp^c(V,\Omega,j)$:

$$\tilde{\Lambda} : U(V,\Omega,j) \to Mp^c(V,\Omega,j) : K \mapsto ((U_K, K) \text{ with parameters } K, 1)$$

i.e. $(U_K f)(z) = f(K^{-1}z)$. At the level of Lie algebras, we have:

$$\tilde{\Lambda}_*(X) = scl(\nu(X)) + \tfrac{1}{2}\mathrm{Trace}_j(X)\mathrm{Id} \qquad \forall X \in \mathfrak{u}(V,\Omega,j).$$

Let $MU^c(V,\Omega,j)$ be the inverse image of $U(V,\Omega,j)$ under σ. It has a character:

$$\lambda : MU^c(V,\Omega,j) \to U(1) : ((U,K)\text{with parameters } K, \tilde{\lambda}) \mapsto \tilde{\lambda}$$

which yields an isomorphism

$$MU^c(V,\Omega,j) \xrightarrow{\sigma\times\lambda} U(V,\Omega,j) \times U(1)$$

with inverse $\tilde{\Lambda} \times i$. Observe that the complex determinant defines another character $\det_j \circ \sigma$ on $MU^c(V,\Omega,j)$and the three characters are related by

$$\eta = \lambda^2 \det_j \circ \sigma.$$

More generally, if $\tilde{j}$ is a compatible complex structure, not necessarily positive, one chooses a positive j commuting with it. The pseudounitary group $U(V,\Omega,\tilde{j})$ of linear endomorphisms injects into $Mp^c(V,\Omega,j)$:

$$U(V,\Omega,\tilde{j}) \to Mp^c(V,\Omega,j) : A \mapsto (U,A) \text{ with param. } ((det_j C_A^-)^{-1}, A)$$

where ${C_A}^-$ is the restriction of C_A to $V_- = \{v \in V \,|\tilde{j}v = -jv\}$.
In fact, if F is any polarization of (V,Ω), the subgroup of symplectic transformations preserving F injects into $Mp^c(V,\Omega,j)$.

3. Mp^{c} structures and Mp^{c} connections

A Mp^c **structure** on (M,ω) is a principal $Mp^{\mathrm{c}}(V,\Omega,j)$ bundle $\mathcal{B}\xrightarrow{p_{\mathcal{B}}} M$ with a fibre-preserving map $\Phi:\mathcal{B}\to\mathcal{B}(M,\omega)$ so that $\Phi(\tilde{f}\cdot\tilde{A})=\Phi(\tilde{f})\cdot\sigma(\tilde{A})$ for all $\tilde{f}\in\mathcal{B}$, $\tilde{A}\in Mp^{\mathrm{c}}(V,\Omega,j)$.

Proposition 1. [2] *Any symplectic manifold (M,ω) admits Mp^c structures, and the isomorphism classes of Mp^c structures on (M,ω) are parametrized by isomorphism classes of complex line bundles.*

An explicit parametrization of Mp^{c} structures is obtained as follows: choose a fibre-wise positive ω-compatible complex structure J on TM; this is always possible since the space of compatible complex structure on a given symplectic vector space is contractible. Define $\mathcal{B}(M,\omega,J)$ to be the principal $U(V,\Omega,j)$ bundle of symplectic frames which are complex linear.
If $(\mathcal{B},\Phi)$ is an Mp^{c} structure, let

$$\mathcal{B}_J:=\Phi^{-1}\mathcal{B}(M,\omega,J).$$

This is a principal $MU^{\mathrm{c}}(V,\Omega,j)\simeq_{\sigma\times\lambda}U(V,\Omega,j)\times U(1)$ bundle and

$$\mathcal{B}\simeq\mathcal{B}_J\times_{MU^{\mathrm{c}}(V,\Omega,j)}Mp^{\mathrm{c}}(V,\Omega,j).$$

Define

$$\mathcal{B}^1_J(\lambda):=\mathcal{B}_J\times_{MU^{\mathrm{c}}(V,\Omega,j),\lambda}U(1)$$

to be the $U(1)$ principal bundle associated to $\mathcal{B}_J$ and to the character λ of $MU^{\mathrm{c}}(V,\Omega,j)$. The map $\tilde{\lambda}:\mathcal{B}_J\to\mathcal{B}^1_J(\lambda):\xi\mapsto[(\xi,1)]$ yields the isomorphism

$$\phi\times\tilde{\lambda}:\mathcal{B}_J\to\mathcal{B}(M,\omega,J)\times_M\mathcal{B}^1_J(\lambda):\xi\mapsto\phi(\xi),[(\xi,1)].$$

The line bundle associated to $\mathcal{B}^1_J(\lambda)$ is denoted by $\mathcal{B}_J(\lambda)$; its isomorphism class is independent of the choice of J.
Reciprocally, given any Hermitean line bundle L over M, one defines the Mp^{c} structure

$$\mathcal{B}:=(\mathcal{B}(M,\omega,J)\times_M L^1)\times_{MU^{\mathrm{c}}(V,\Omega,j)}Mp^{\mathrm{c}}(V,\Omega,j).$$

A Mp^{c}**-connection** on the Mp^{c} structure $(\mathcal{B},\Phi)$ is a principal connection α on $\mathcal{B}$; in particular, it is a 1-form on $\mathcal{B}$ with values in $\mathfrak{mp}^{\mathrm{c}}\simeq_{\sigma_*\times\eta_*}\mathfrak{sp}(V,\Omega))\oplus\mathfrak{u}(1)$. We decompose it accordingly as

$$\alpha=\alpha_1+\alpha_0.$$

The character η yields the construction of a $U(1)$ principal bundle

$$\mathcal{B}^1(\eta):=\mathcal{B}\times_{Mp^{\mathrm{c}}(V,\Omega,j),\eta}U(1)$$

and there exists a map

$$\tilde{\eta}:\mathcal{B}\to\mathcal{B}^1(\eta):\xi\mapsto[\xi,1].$$

Then α_0 is the pull-back of a $\mathfrak{u}(1)$-valued 1-form on $\mathcal{B}^1(\eta)$ under the differential of $\tilde{\eta}$ and

$$\alpha_0=2\tilde{\eta}^*\beta_0$$

where β_0 is a principal $U(1)$ connection on $\mathcal{B}^1(\eta)$.
Similarly α_1 is the pull-back under the differential of $\Phi : \mathcal{B} \to \mathcal{B}(M,\omega)$ of a $\mathfrak{sp}(V,\Omega)$-valued 1-form β_1 on $\mathcal{B}(M,\omega)$ and

$$\alpha_1 = \Phi^* \beta_1$$

where β_1 is a principal $Sp(V,\Omega)$ connection on $\mathcal{B}(M,\omega)$, hence corresponding to a linear connection ∇ on M so that $\nabla\omega = 0$.
Thus a Mp^c-connection on $\mathcal{B}$ induces connections in TM preserving ω and in $\mathcal{B}^1(\eta)$. The converse is true – we pull back and add connection 1-forms in $\mathcal{B}^1(\eta)$ (with a factor 2) and in $\mathcal{B}(M,\omega)$ to get a connection 1-form on $\mathcal{B}$.

In geometric quantization, a **prequantization structure** is a Mp^c-struture $(\mathcal{B},\Phi)$ with a connection α so that

$$d\alpha_0 = \pi^* \frac{\omega}{i\hbar}.$$

The prequantization module is the module of sections of symplectic spinors $\mathcal{B} \times_{Mp^c(V,\Omega,j)} \mathcal{H}^{-\infty}(V,\Omega,j)$ with prequantisation operators

$$Q(f) := \nabla^{\alpha}_{X_f} + \frac{1}{i\hbar} f.$$

We refer to [2] for further development and quantization in this context (which yields automatically half-forms).
The condition for the existence of such a structure is that

$$[\frac{\omega}{h}] - \tfrac{1}{2} c_1(TM,\omega)^{\mathbb{R}} \tag{5}$$

be an integral cohomology class, where $c_1(TM,\omega)$ is the first Chern class of the tangent bundle viewed as a complex bundle via the choice of a compatible complex structure. Indeed $c_1(TM,\omega)$ is the first Chern class of the line bundle associated to $\mathcal{B}(M,\omega,J)$ and the character $\det_{j\circ}\sigma$; since $\eta = \lambda^2 \det_{j\circ}\sigma$, we have

$$c_1(\mathcal{B}(\eta)) = 2c_1(\mathcal{B}_J(\lambda)) + c_1(TM,\omega),$$

but $c_1(\mathcal{B}(\eta)) = [\frac{i}{2\pi} d\beta_0]$ and $c_1(\mathcal{B}_J(\lambda))$ must be an integral class.
Condition (5) was obtained in the context of Deformation Quantization by Boris Fedosov [6] in his construction of asymptotic operator representations of a star product on a symplectic manifold.A geometric interpretation of this result using Mp^c-structures is given in [7].

4. Dirac operators

Given a $Spin^c$ structure $(\mathcal{B},\Phi)$ on an oriented Riemannian manifold (M,g) one considers the spinor bundle

$$\mathcal{S} = \mathcal{B} \times_{(Spin^c,r)} S.$$

Since the representation r on S is unitary, the spinor bundle carries a natural fiberwise Hermitian inner product, still denoted $<,>$.

The smooth sections of $\mathcal{S}$ are called **spinor fields** on M. For two spinor fields ψ, ψ' with compact support on M, one defines their inner product

$$<< \psi, \psi' >>:= \int_M < \psi(x), \psi'(x) > d^g x$$

where $d^g x$ is the measure on M associated to the metric g.
The Clifford multiplication of TM on $\mathcal{S}$ given by

$$Cl([(\tilde{f}, v)]) \Big([(\tilde{f}, s)] \Big) := [(\tilde{f}, cl(v)s)] \qquad \forall \tilde{f} \in \mathcal{B}, v \in V, s \in S$$

yields a map

$$Cl : \Gamma(M, TM) \times \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}) : (X, \psi) \mapsto Cl(X)\psi.$$

A $Spin^c$ **connection** on the $Spin^c$ structure $(\mathcal{B}, \Phi)$ is a principal connection α on $\mathcal{B}$; in particular, it is a 1-form on $\mathcal{B}$ with values in $\mathfrak{spin}^c \simeq \mathfrak{so}(V, \Omega)) \oplus \mathfrak{u}(1)$. We decompose it accordingly as

$$\alpha = \alpha_1 + \alpha_0.$$

The 1-form α_1 is the pull-back under the differential of $\Phi : \mathcal{B} \to \mathcal{B}(M, \tilde{g})$ of a $\mathfrak{so}(V, \tilde{g})$-valued 1-form β_1 on $\mathcal{B}(M, g)$

$$\alpha_1 = \Phi^* \beta_1$$

and β_1 is a principal $SO(V, \tilde{g})$ connection on $\mathcal{B}(M, g)$, hence corresponding to a linear connexion ∇ on M such that $\nabla g = 0$. We choose this to be the Levi Civita connection.
A $Spin^c$ connection α induces a covariant derivative ∇^α of the spinor fields :

$$\nabla^\alpha : \Gamma(M, \mathcal{S}) \to \Gamma(M, T^*M \otimes \mathcal{S}) : \psi \mapsto [X \to \nabla^\alpha_X \psi]$$

and the Clifford multiplication is parallel

$$\nabla^\alpha_X \left(Cl(Y)\psi \right) = Cl(\nabla_X Y)\psi + Cl(X)\nabla^\alpha_X \psi.$$

The $Spin^c$ **Dirac operator** is the differential operator of order 1 acting on spinor fields given by the contraction of the covariant derivative and the Clifford multiplication, using concatenation and the identification of TM and T^*M induced by g:

$$D\psi := \sum_i Cl(e_i) \nabla^\alpha_{e^i} \psi = \sum_{ij} g^{ij} Cl(e_i) \nabla^\alpha_{e_j} \psi$$

where e_i is a local frame field for TM and e^i the dual frame field defined by $g(e_i, e^j) = \delta_i^j$.
This Dirac operator D is elliptic and selfadjoint. It acts on the sections of a finite dimensional bundle. Its square is equal to

$$D^2\psi = -\sum_{jl} g^{jl} \nabla^{\alpha 2}_{jl} \psi + \tfrac{1}{2} \sum_{ijkl} g^{ij} g^{kl} cl(e_i) cl(e_k) \left(R^\alpha(e_j, e_l)\psi - \nabla_{T^\alpha(e_j, e_l)} \psi \right)$$

where R^α is the curvature and T^α the torsion of ∇^α acting on $\mathcal{S}$; with our choice that α_1 corresponds to the Levi Civita connection on M the torsion T^α vanishes.

The **classical Dirac operator** is defined similarly using a *Spin* structure $(\mathcal{B}, \Phi)$. Since σ_* yields an isomophism of Lie algebras between $\mathfrak{spin}$ and $\mathfrak{so}(V, \tilde{g})$, a natural *Spin*-connection α' on $\mathcal{B}$ is given by

$$\alpha' = \sigma_*^{-1}\Phi^*\beta_1$$

where β_1 is the principal $SO(V, \tilde{g})$ connection on $\mathcal{B}(M, g)$ defined by the Levi Civita connection on (M, g).
For further results about the $Spin^c$ Dirac operator and its relation to the classical Dirac operator, we refer to [8].

Given a Mp^c structure $(\mathcal{B}, \Phi)$ on a symplectic manifold (M, ω) one considers the symplectic spinor bundle

$$\mathcal{S} = \mathcal{B} \times_{(Mp^c, r)} \mathcal{H}^{\pm\infty}.$$

Since the representation r on $\mathcal{H}$ is unitary and preserves $\mathcal{H}^\infty$, the spinor bundle carries a fiberwise Hermitian inner product, still denoted $(\ ,\ )_j$.
The smooth sections of $\mathcal{S}$ are called **symplectic spinor fields** on M. For two symplectic spinor fields ψ, ψ' with compact support on M, one defines their inner product

$$<< \psi, \psi' >>_j := \int_M (\psi(x), \psi'(x))_j d^\omega x$$

where $d^\omega x$ is the measure on M associated to the symplectic form ω.
The symplectic Clifford multiplication of TM on $\mathcal{S}$ given by

$$sCl([(\tilde{f}, v)]) \left([(\tilde{f}, s)]\right) := [(\tilde{f}, scl(v)s)] \qquad \forall \tilde{f} \in \mathcal{B}, v \in V, s \in \mathcal{H}^{\pm\infty}$$

yields a map

$$sCl : \Gamma(M, TM) \times \Gamma(M, \mathcal{S}) \to \Gamma(M, \mathcal{S}) : (X, \psi) \mapsto sCl(X)\psi.$$

A Mp^c connection α induces a covariant derivative ∇^α of the symplectic spinor fields :

$$\nabla^\alpha : \Gamma(M, \mathcal{S}) \to \Gamma(M, T^*M \otimes \mathcal{S}) : \psi \mapsto [X \to \nabla^\alpha_X \psi]$$

and the Clifford multiplication is parallel

$$\nabla^\alpha_X (sCl(Y)\psi) = sCl(\nabla_X Y)\psi + sCl(X)\nabla^\alpha_X\psi.$$

The Mp^c **symplectic Dirac operator** is the differential operator of order 1 acting on symplectic spinor fields given by the contraction of the covariant derivative and the symplectic Clifford multiplication, using concatenation and the identification of TM and T^*M induced by ω:

$$D\psi := \sum_i sCl(e_i)\nabla^\alpha_{e^i}\psi = -\sum_{ij} \omega^{ij} sCl(e_i)\nabla^\alpha_{e_j}\psi$$

where e_i is a local frame field for TM and e^i is the dual frame field defined by $\omega(e_i, e^j) = \delta_i^j$.

The definition of the Dirac operator in the Mp^c framework given above is a straightforward generalisation of the **symplectic Dirac operator** studied by Katharina and Lutz Habermann [5, 9, 10]. They use a **metaplectic structure** $(\mathcal{B}, \Phi)$, i.e. a $Mp(V, \Omega, j)$- principal bundle $\mathcal{B}$ and a fiberpreserving

$(Mp(V,\Omega,j),\sigma)$ equivariant map $\Phi : \mathcal{B} \to \mathcal{B}(M,\omega)$. Remark that those do not always exist.
Since σ_* is an isomophism of Lie algebras between $\mathfrak{mp}(V,\Omega,j)$ and $\mathfrak{sp}(V,\Omega)$, a metaplectic connection α' on $\mathcal{B}$ is given by

$$\alpha' = \sigma_*^{-1}\Phi^*\beta_1$$

where β_1 is a principal $Sp(V,\Omega)$ connection on $\mathcal{B}(M,\omega)$ defined by a linear connection ∇ on (M,ω) so that $\nabla\omega = 0$.
K. Habermann introduces a second symplectic Dirac operator $\tilde{D}$ using an auxiliary compatible almost complex struture J on (M,ω):

$$\widetilde{D}\psi := \sum_{ij} g^{ij} Cl(e_i)\nabla_{e_j}\psi \quad \text{with} \quad g(X,Y) := \omega(X,JY)$$

using a connection so that ω and J are parallel.
The commutator of D and $\tilde{D}$ is elliptic and acts on sections of an infinite family of finite dimensional subbundles of $\mathcal{S}$.
This construction can be performed in the Mp^{c} framework [3]. It can be further generalised, also in the Riemannian context, using some fields of endomorphisms of the tangent bundle which are covariantly constant. We give a brief description below.

We consider a field A of endomorphisms of the tangent bundle TM of an oriented Riemannian manifold (M,g), such that

$$g(AX,Y) = \epsilon_A g(X,AY) \quad \text{with } \epsilon_A = \pm 1, \qquad \forall X,Y \in \Gamma(M,TM),$$

and such that there is a linear connection ∇ preserving g and A, i.e.

$$\nabla g = 0 \qquad \text{and} \qquad \nabla A = 0.$$

Given a $Spin^{\mathrm{c}}$ structure $(\mathcal{B},\Phi)$ on (M,g), we consider a $Spin^{\mathrm{c}}$ connection $\alpha = \alpha_1 + \alpha_0$ so that $\alpha_1 = \Phi^*\beta_1$ where β_1 is the connection 1-form on $\mathcal{B}(M,g)$ defined by the linear connection ∇. We define a new Dirac operator D_A :

$$D_A\psi := \sum_i Cl(Ae_i)\nabla^\alpha_{e^i}\psi = \sum_{ij} A_i^k g^{ij} Cl(e_k)\nabla^\alpha_{e_j}\psi$$

where e_i is a local frame field for TM and e^i is defined by $g(e_i,e^j) = \delta_i^j$.
Remark that the $Spin^{\mathrm{c}}$ Dirac operator corresponds to $A = \mathrm{Id}$ with $\epsilon_A = 1$.
On a Kähler manifold (M,g,J), one also has D_J with $\epsilon_J = -1$. We have

$$\begin{aligned} D_A^2\psi = &-\epsilon_A \sum_{jst} (A^2)_s^j\, g^{st}\, \nabla^{\alpha 2}_{e_j e_t}\psi \\ &+ \tfrac{1}{2} \sum_{ijkrst} A_i^k A_s^r g^{ij} g^{st} Cl(e_k)Cl(e_r) \left(R^\alpha(e_j,e_t)\psi - \nabla^\alpha_{T^\alpha(e_j,e_t)}\psi\right). \end{aligned}$$

More generally, if A and B are two such fields of endomorphisms, we have

$$(D_A \circ D_B + D_B \circ D_A)\psi = -\sum_{jst} (\epsilon_A (AB)^j_s + \epsilon_B (BA)^j_s)\, g^{st}\, \nabla^{\alpha 2}_{e_j e_t}\psi$$

$$+ \tfrac{1}{2} \sum_{ijkrst} (A^k_i B^r_s + A^r_s B^k_i) g^{ij} g^{st} Cl(e_k) Cl(e_r) \left(R^\alpha(e_j, e_t)\psi - \nabla^\alpha_{T^\alpha(e_j,e_t)}\psi \right).$$

Similarly we consider a field A of endomorphisms of the tangent bundle TM of a symplectic manifold (M, ω), such that

$$\omega(AX, Y) = \epsilon_A \omega(X, AY) \quad \text{with } \epsilon_A = \pm 1, \qquad \forall X, Y \in \Gamma(M, TM),$$

and such that there is a linear connection ∇ preserving ω and A, i.e.

$$\nabla\omega = 0 \qquad \text{and} \qquad \nabla A = 0.$$

Given a Mp^{c} structure $(\mathcal{B}, \Phi)$ on (M, ω), we consider a Mp^{c} connection $\alpha = \alpha_1 + \alpha_0$ so that $\alpha_1 = \Phi^* \beta_1$ where β_1 is the connection 1-form on $\mathcal{B}(M, \omega)$ defined by the linear connection ∇. We define a new Dirac operator D_A :

$$D_A\psi := \sum_i sCl(Ae_i)\nabla^\alpha_{e^i}\psi = \sum_{ij} A^k_i \omega^{ij} sCl(e_k)\nabla^\alpha_{e_j}\psi$$

where e_i is a local frame field for TM and e^i is defined by $\omega(e_i, e^j) = \delta^j_i$. The Mp^{c} Dirac operator corresponds to $A = \mathrm{Id}$ with $\epsilon_A = 1$. The operator $\tilde{D}$ is equal to D_J for J a positive compatible complex struture on (M, ω); then $\epsilon_J = -1$ and there always exists a linear connection preserving ω and J; it may have torsion, but one can always assume that the torsion vector vanishes, i.e. $\sum_i T^\alpha(e_i, e^i) = 0$. If A and B are two such fields of endomorphisms, we have

$$(D_A \circ D_B - D_B \circ D_A)\psi = -\frac{i}{2\hbar} \sum_{jst} (\epsilon_A (AB)^j_s - \epsilon_B (BA)^j_s)\, \omega^{st}\, \nabla^{\alpha 2}_{e_j e_t}\psi$$

$$+ \tfrac{1}{2} \sum_{ijkrst} (A^k_i B^r_s - A^r_s B^k_i) \omega^{ij} \omega^{st} sCl(e_k) sCl(e_r) \left(R^\alpha(e_j, e_t)\psi - \nabla^\alpha_{T^\alpha(e_j,e_t)}\psi \right).$$

These generalized symplectic Dirac operators and their commutators are particularly relevant in a homogeneous or symmetric framework, for instance when there is a homogeneous non positive definite compatible almost complex structure on (M, ω) or two homogeneous distributions of supplementary real lagrangians; this is work in progress with Laurent La Fuente Gravy and John Rawnsley.

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Michel Cahen
Université Libre de Bruxelles, Campus Plaine, CP 218
Bvd du triomphe,1050 Bruxelles, Belgique
e-mail: `mcahen@ulb.ac.be`

Simone Gutt
Université Libre de Bruxelles, Campus Plaine, CP 218
Bvd du triomphe,1050 Bruxelles, Belgique
and
Université de Lorraine à Metz
Ile du Saulcy, F-57045 Metz Cedex 01, France
e-mail: `sgutt@ulb.ac.be`